2007 Paper 2 Question 6

Discrete Mathematics II

(*a*) Let V be a set of propositional variables, and let $\mathcal{F}_V$ be the set of propositional formulae (or Boolean propositions) with propositional variables in V.

Consider the set of rule instances R given by

Axiom K $$\frac{}{A \Rightarrow (B \Rightarrow A)}$$

Axiom S $$\frac{}{(A \Rightarrow (B \Rightarrow C)) \Rightarrow \big((A \Rightarrow B) \Rightarrow (A \Rightarrow C)\big)}$$

Axiom C $$\frac{}{(\neg A \Rightarrow \neg B) \Rightarrow (B \Rightarrow A)}$$

Rule MP $$\frac{A \qquad A \Rightarrow B}{B}$$

where $A, B, C \in \mathcal{F}_V$.

As usual, let $\mathcal{I}_R \subseteq \mathcal{F}_V$ denote the set inductively defined by R (that is, the smallest R-closed set). Furthermore, let $\mathcal{T} \subseteq \mathcal{F}_V$ be the set of tautologies.

You are required to establish that $\mathcal{I}_R \subseteq \mathcal{T}$, by rule induction. Specifically, state precisely what needs to be proved with respect to each of the Axioms K, S, C and the Rule MP; but only give details of the proofs associated to Axiom C and Rule MP. [12 marks]

[One can also show that $\mathcal{T} \subseteq \mathcal{I}_R$; but this is outside the scope of the question.]

(*b*) The purpose of this part of the question is to study diagonalisation, or Cantor's diagonal argument, and one of its consequences.

For sets X and Y, let $(X \to Y)$ denote the set of all functions from X to Y.

(*i*) A function $f : X \to X$ is said to have a fixed point if there exists an element $x \in X$ such that $f(x) = x$; an element with this property is called a *fixed point* of the function.

Prove the following *Diagonalisation Theorem*: For sets N and X, if there exists a surjection $N \twoheadrightarrow (N \to X)$ then every function $X \to X$ has a fixed point. [4 marks]

[Hint: Let $e : N \twoheadrightarrow (N \to X)$ be a surjection and, for $f : X \to X$, consider the function $\varphi : N \to X$ defined by $\varphi(n) \stackrel{\text{def}}{=} f\big(e(n)(n)\big)$, for all $n \in N$.]

(*ii*) Using the Diagonalisation Theorem, or otherwise, show that if there exists a surjection $D \twoheadrightarrow (D \to D)$, for a set D, then D has exactly one element. [4 marks]