2005 Paper 12 Question 9

Numerical Analysis II

The best L_∞ approximation to $f(x) \in C[-1,1]$ by a polynomial $p_{n-1}(x)$ of degree $n-1$ has the property that

$$\max_{x \in [-1,1]} |e(x)|$$

is attained at $n+1$ distinct points $-1 \le \xi_0 < \xi_1 < \ldots < \xi_n \le 1$ such that $e(\xi_j) = -e(\xi_{j-1})$ for $j = 1, 2, \ldots n$ where $e(x) = f(x) - p_{n-1}(x)$.

(a) Let $f(x) = x^2$. Show, by means of a clearly labelled sketch graph, that the best polynomial approximation of degree 1 is a constant. [3 marks]

(b) Now suppose $f(x) = (x+1)/(x+\frac{5}{3})$ is the function to be approximated over $[-1,1]$. By sketching the graph, deduce properties of the best linear approximation $p_1(x)$. By differentiating $e(x)$, find $p_1(x)$. [9 marks]

(c) Now consider $f(x) = x/(9x^2+16)$. Explain why the best approximation over $[-1,1]$ of degree 2 or less is of the form $p_2(x) = ax$, and sketch the graph to show the extreme values of $e(x)$. Verify that $x = 4/9$ is one of the extreme values and find a. [8 marks]