2005 Paper 11 Question 10

Computation Theory

(*a*) What does it mean for a subset S of the set $\mathbb{N}$ of natural numbers to be register machine *decidable*? [3 marks]

(*b*) For each $e \in \mathbb{N}$, let $\varphi_e \in \mathit{Pfn}(\mathbb{N}, \mathbb{N})$ denote the partial function computed by the register machine program with index e. Let $e_0 \in \mathbb{N}$ be an index for the totally undefined partial function (so that $\varphi_{e_0}(x)\uparrow$, for all $x \in \mathbb{N}$).

Suppose that a total function $f \in \mathit{Fun}(\mathbb{N}, \mathbb{N})$ is *extensional*, in the sense that for all $e, e' \in \mathbb{N}$, $f(e) = f(e')$ if φ_e and $\varphi_{e'}$ are equal partial functions. Suppose also that the set $S_f = \{x \in \mathbb{N} \mid f(x) = f(e_0)\}$ is not the whole of $\mathbb{N}$, so that for some $e_1 \in \mathbb{N}$, $f(e_1) \neq f(e_0)$.

(*i*) If membership of S_f were decided by a register machine M, show informally how to construct from M a register machine M' that, started with $R1 = e$ and $R2 = n$ (any $e, n \in \mathbb{N}$) always halts, with $R0 = 0$ if $\varphi_e(n)\downarrow$ and with $R0 = 1$ if $\varphi_e(n)\uparrow$. Make clear in your argument where you use the fact that f is extensional.

[Hint: For each $e, n \in \mathbb{N}$ consider the index $i(e, n) \in \mathbb{N}$ of the register machine that inputs x, computes $\varphi_e(n)$ and if that computation halts, then computes $\varphi_{e_1}(x)$.] [14 marks]

(*ii*) Deduce that if f is extensional, then S_f is either the whole of $\mathbb{N}$, or not decidable. [3 marks]