2004 Paper 9 Question 15

Denotational Semantics

(*a*) Show that any continuous function $h : D \to D$ on a domain D has a least prefixed point $\mathit{fix}(h)$. [5 marks]

(*b*) Let $f : D \times E \to D$ and $g : D \times E \to E$ be continuous functions where D and E are domains. The continuous function $\langle f, g\rangle : D \times E \to D \times E$, acts so that $(d, e) \mapsto (f(d, e), g(d, e))$. Bekič's theorem states that the least fixed point of $\langle f, g\rangle$ is the pair $(\hat{d}, \hat{e})$ where

$$\begin{aligned} \hat{d} &= \mathit{fix}(\lambda d.\ f(d, \hat{e})) \text{ where} \\ \hat{e} &= \mathit{fix}(\lambda e.\ g(\mathit{fix}(\lambda d.\ f(d, e)), e)) \ . \end{aligned}$$

You are asked to show Bekič's theorem in the following stages. Write (d_0, e_0) for the least fixed point of $\langle f, g\rangle$.

(*i*) Show that $(\hat{d}, \hat{e})$ is a fixed point of $\langle f, g\rangle$. Deduce that $(d_0, e_0) \sqsubseteq (\hat{d}, \hat{e})$. [5 marks]

(*ii*) Show the converse, that $(\hat{d}, \hat{e}) \sqsubseteq (d_0, e_0)$. [10 marks]