2004 Paper 4 Question 6

Continuous Mathematics

For non-negative integers r and s we have the orthogonality properties

$$\int_0^{2\pi} \cos(rx)\cos(sx)dx = \begin{cases} 2\pi & \text{if } r = s = 0 \\ \pi\delta_{rs} & \text{otherwise} \end{cases}$$

$$\int_0^{2\pi} \sin(rx)\sin(sx)dx = \begin{cases} 0 & \text{if } r = s = 0 \\ \pi\delta_{rs} & \text{otherwise} \end{cases}$$

$$\int_0^{2\pi} \sin(rx)\cos(sx)dx = 0 \qquad \forall \quad r, s$$

where

$$\delta_{rs} = \begin{cases} 1 & \text{if } r = s \\ 0 & \text{otherwise} \end{cases} .$$

(*a*) Derive expressions for the Fourier coefficients a_0, a_n, b_n $(n = 1, 2, \ldots)$ such that the infinite series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

is the Fourier series for the function $f(x)$ in an interval of length 2π. [6 marks]

(*b*) For any fixed integer $N \geq 1$ let

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N-1} (a_n \cos(nx) + b_n \sin(nx))$$

be the Fourier series for $f(x)$ truncated to the first N terms and let

$$S'_N(x) = \frac{a'_0}{2} + \sum_{n=1}^{N-1} (a'_n \cos(nx) + b'_n \sin(nx))$$

be any other Fourier series truncated to the first N terms. Show that

$$\int_0^{2\pi} (f(x) - S_N(x))\,(S_N(x) - S'_N(x))\,dx = 0\,.$$

[8 marks]

(*c*) Given the function $f(x)$ show that

$$\int_0^{2\pi} (f(x) - S'_N(x))^2\,dx$$

is minimised by the unique choice $a'_0 = a_0$, $a'_n = a_n$, $b'_n = b_n$ $(n = 1, 2, \ldots)$, that is, the Fourier series gives the best approximation to $f(x)$ using N terms in the sense of minimising the mean-squared error. [6 marks]