2004 Paper 13 Question 9

Numerical Analysis II

(a) A Riemann integral over $[a, b]$ is defined by

$$\int_a^b f(x)\,dx = \lim_{\substack{n\to\infty \\ \Delta\xi\to 0}} \sum_{i=1}^{n} (\xi_i - \xi_{i-1}) f(x_i) \ .$$

Explain the terms *Riemann sum* and *mesh norm.* [4 marks]

(b) Consider the quadrature rule

$$Qf = \frac{3h}{8}[f(a) + 3f(a+h) + 3f(a+2h) + f(a+3h)] - \frac{3f^{(4)}(\lambda)h^5}{80} \ .$$

If $[a, b] = [-1, 1]$ find $\xi_0, \xi_1, \ldots, \xi_4$ and hence show that this is a Riemann sum. [3 marks]

(c) Suppose R is a rule that integrates constants exactly over $[-1, 1]$, and that $f(x)$ is bounded and Riemann-integrable over $[a, b]$. Write down a formula for the composite rule $(n \times R)f$ and prove that

$$\lim_{n\to\infty} (n \times R)f = \int_a^b f(x)\,dx$$ [6 marks]

(d) What is the formula for $(n \times Q)f$ over $[a, b]$? [4 marks]

(e) Which polynomials are integrated exactly by Qf? Which monomials are integrated exactly by the product rule $(Q \times Q)F$ when applied to a function of x and y? [3 marks]