2003 Paper 1 Question 7

Discrete Mathematics

(a) Prove that there are infinitely many prime numbers. [4 marks]

(b) Let $p_1, p_2, \ldots, p_k$ be the first k primes. Show that the number of positive integers less than n and having no prime factors other than $p_1, p_2, \ldots, p_k$ is less than $\sqrt{n}2^k$. [8 marks]

[Hint: All such numbers are of the form $m^2\, p_1^{\varepsilon_1}\, p_2^{\varepsilon_2} \cdots p_k^{\varepsilon_k}$ where each ε_i is 0 or 1.]

Deduce that the k^{th} prime is less than 4^k. [4 marks]

(c) State the inclusion–exclusion principle and use it to determine the number of prime numbers less than 100. [4 marks]