2001 Paper 13 Question 10

Numerical Analysis II

(*a*) Taylor's theorem states that if $x \in [a, b]$ and $f \in C^{N+1}[a, b]$

$$f(x) = T_N(a) + \frac{1}{N!} \int_a^x f^{(N+1)}(t)(x-t)^N \, dt$$

where

$$T_N(a) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^N}{N!} f^{(N)}(a).$$

Prove Taylor's theorem. [6 marks]

(*b*) Peano's theorem states that if a quadrature rule integrates polynomials of degree N exactly over an interval $[a, b]$ then the error in integrating $f \in C^{N+1}[a, b]$ can be expressed as

$$E(f) = \int_a^b f^{(N+1)}(t) K(t) \, dt$$

where

$$K(t) = \frac{1}{N!} E_x[(x-t)_+^N].$$

Explain the notation $E(f)$, E_x and $(x-t)_+^N$. [4 marks]

(*c*) Use Taylor's theorem to prove Peano's theorem. [8 marks]

(*d*) Under what additional condition may the simplified formula

$$E(f) = \frac{f^{(N+1)}(\xi)}{(N+1)!} E(x^{N+1})$$

be applied? [2 marks]