2001 Paper 10 Question 11

Mathematics for Computation Theory

(*a*) Define precisely what is meant by the following:

(*i*) $\prec$ is a well-founded relation on the set S;

(*ii*) $y \in S$ is a minimal element for $\prec$. [3 marks]

(*b*) If $\prec$ is a well-founded relation on S, show that every non-empty subset of S contains an element that is minimal for $\prec$. [4 marks]

(*c*) Let $(P, \leqslant)$ be a finite partially ordered set. A *chain* $X \subseteq P$ is a totally ordered subset of P, and an *antichain* $Y \subseteq P$ is a subset such that no two distinct elements y, $y' \in Y$ are comparable. The antichains $\{Y_i \mid 1 \leqslant i \leqslant k\}$ cover P if $P \subseteq \bigcup_{i=1}^{k} Y_i$.

Prove that the smallest possible number of antichains in a cover of P is exactly the length of a longest chain in P. [Hint: If not, consider the set of minimal elements in a minimal counterexample.]

[13 marks]