2000 Paper 4 Question 2

Continuous Mathematics

(*a*) In his formulation of the calculus, Newton captured only the notion of integer-order differentiation considering first-, second- and third-order derivatives, and so on. In scientific computing, however, we sometimes need fractional-order derivatives, as for example in fluid mechanics.

Explain how *Fractional Differentiation* (derivatives of non-integer order) can be given precise quantitative meaning through Fourier analysis. [5 marks]

Suppose that a continuous function $f(x)$ has Fourier Transform $F(\mu)$. Outline an algorithm (as a sequence of mathematical steps, not an actual program) for computing the 1.5^{th} derivative of some function $f(x)$

$$\frac{d^{(1.5)} f(x)}{dx^{(1.5)}}$$

[5 marks]

(*b*) For $i = \sqrt{-1}$, consider the quantity $\sqrt{i}$.

(*i*) Express $\sqrt{i}$ as a complex exponential. [1 mark]

(*ii*) In which quadrant of the complex plane does it lie? [1 mark]

(*iii*) What is the real part of $\sqrt{i}$? [1 mark]

(*iv*) What is the imaginary part of $\sqrt{i}$? [1 mark]

(*v*) What is the length (the modulus) of $\sqrt{i}$? [1 mark]

(*c*) Initial-value problems described by ordinary differential equations have solutions that can be propagated forward using incrementing rules such as Euler or Runge–Kutta. But boundary-value problems specified by partial differential equations (PDEs) such as Poisson's Equation,

$$\frac{\partial^2 \mu(x,y)}{\partial x^2} + \frac{\partial^2 \mu(x,y)}{\partial y^2} = \rho(x,y)$$

cannot be solved by such propagation methods. Why not? [3 marks]

State the principle for one general class of numerical methods for solving such PDEs. [2 marks]