2000 Paper 1 Question 2

Discrete Mathematics

State the conditions for a relation to be a partial order. [3 marks]

A *partition* of a natural number n is a collection of natural numbers (possibly including duplicates and in any order) whose sum is n. Let P_n be the set of partitions of n; for example, $P_4 = \{(4), (3,1), (2,2), (2,1,1), (1,1,1,1)\}$. Order the partitions in P_n as follows:

$(a_1, a_2, \ldots a_r) \leqslant (b_1, b_2, \ldots b_s)$ if the (a_i) and (b_j) can be rearranged so that

$$\begin{aligned}
b_1 &= a_1 + a_2 + \cdots + a_{k_1} \\
b_2 &= a_{k_1+1} + a_{k_1+2} + \cdots + a_{k_2} \\
&\vdots \\
b_{s-1} &= a_{k_{s-2}+1} + a_{k_{s-2}+2} + \cdots + a_{k_{s-1}} \\
b_s &= a_{k_{s-1}+1} + a_{k_{s-1}+2} + \cdots + a_r
\end{aligned}$$

Note that $(2,1,1) \leqslant (3,1)$, and $(2,1,1) \leqslant (2,2)$ but $(3,1)$ and $(2,2)$ cannot be compared.

Show that $\leqslant$ is a partial order on P_n. [4 marks]

P_5 has seven elements; draw the Hasse diagram for $(P_5, \leqslant)$. [3 marks]