1998 Paper 7 Question 10

Types

Each natural number $n \in \mathbb{N}$ can be encoded in the polymorphic lambda calculus (PLC) by the beta-normal form $Num_n \stackrel{\text{def}}{=} \Lambda\,\alpha\,(\lambda\,x : \alpha\,(\lambda\,f : \alpha \to \alpha\,(f^n x)))$, where the expression $f^n x$ is an abbreviation for the PLC expression inductively defined by

$$f^0 x \stackrel{\text{def}}{=} x \qquad \text{and} \qquad f^{n+1} x \stackrel{\text{def}}{=} f\,(f^n x).$$

For which PLC type nat does $\vdash Num_n : nat$ hold? [3 marks]

Say that a function $\phi : \mathbb{N} \to \mathbb{N}$ is *PLC-representable* if there is a PLC expression F such that the following typing and beta-conversions hold:

$$\vdash F : nat \to nat \qquad \text{and} \qquad F\,Num_n =_\beta Num_{\phi(n)} \quad (\text{all } n \in \mathbb{N}).$$

Show that the successor function, $s(n) \stackrel{\text{def}}{=} n + 1$, is PLC-representable. [5 marks]

Given a PLC-representable function ϕ and a number $a \in \mathbb{N}$, show that the function ψ inductively defined by

$$\psi(0) \stackrel{\text{def}}{=} a \qquad \text{and} \qquad \psi(n+1) \stackrel{\text{def}}{=} \phi(\psi(n))$$

is PLC-representable. [8 marks]

Is every function $\mathbb{N} \to \mathbb{N}$ PLC-representable? Justify your answer. [4 marks]