1998 Paper 6 Question 9

Foundations of Functional Programming

The binary trees, denoted by B, whose branch nodes contain natural numbers, are generated by the grammar

$$B ::= \mathsf{Leaf} \mid \mathsf{Br}\ (n, B, B)$$

where n ranges over natural numbers. Although this question concerns the encoding of binary trees as λ-terms, you may use the encodings of other well-known data structures, such as booleans and pairs, provided you state the properties assumed.

Give an encoding of binary trees as λ-terms by defining as λ-terms

(*a*) **Leaf** and **Br**, used to construct the λ-terms corresponding to binary trees;

(*b*) **isLeaf**, which tests whether a λ-term corresponds to a leaf or a branch node;

(*c*) **value**, **fstsubtree** and **sndsubtree**, used to identify respectively the natural number and the two subtrees at a branch node.

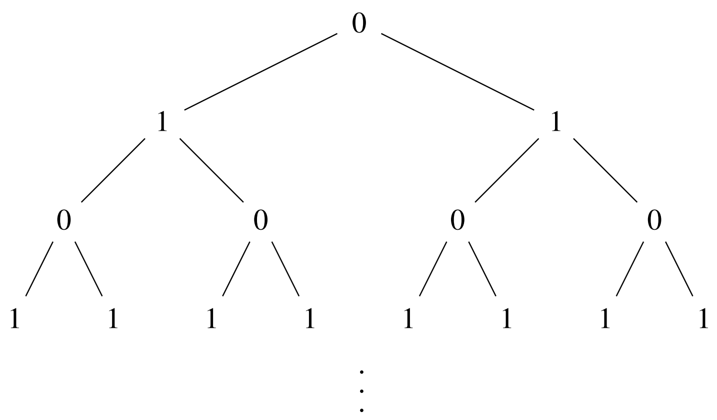
Justify your answer by describing the behaviour of **isLeaf**, **value**, **fstsubtree** and **sndsubtree**: for example, the reduction **isLeaf**(**Leaf**) $\twoheadrightarrow$ **true** describes part of the behaviour of **isLeaf**. [8 marks]

Consider the function $\mathsf{treeadd}$ defined inductively on the structure of binary trees by

$$\begin{aligned} \mathsf{treeadd}\ (m, \mathsf{Leaf}) &= \mathsf{Leaf} \\ \mathsf{treeadd}\ (m, \mathsf{Br}\ (n, B_1, B_2)) &= \mathsf{Br}\ (m+n, \mathsf{treeadd}\ (m, B_1), \mathsf{treeadd}\ (m, B_2)) \end{aligned}$$

Give and justify a λ-term which encodes $\mathsf{treeadd}$, using the λ-term $Y \equiv \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$. [6 marks]

Give the λ-term for the infinite binary tree whose branch nodes consist of zeros at even depths and ones at odd depths, as pictured below:



[6 marks]