Foundations of Functional Programming

The binary trees, denoted by $B$, whose branch nodes contain natural numbers, are generated by the grammar

$$B ::= \text{Leaf} \mid \text{Br}(n, B, B)$$

where $n$ ranges over natural numbers. Although this question concerns the encoding of binary trees as $\lambda$-terms, you may use the encodings of other well-known data structures, such as booleans and pairs, provided you state the properties assumed.

Give an encoding of binary trees as $\lambda$-terms by defining as $\lambda$-terms

(a) $\text{Leaf}$ and $\text{Br}$, used to construct the $\lambda$-terms corresponding to binary trees;

(b) $\text{isLeaf}$, which tests whether a $\lambda$-term corresponds to a leaf or a branch node;

(c) $\text{value}$, $\text{fstsubtree}$ and $\text{sndsubtree}$, used to identify respectively the natural number and the two subtrees at a branch node.

Justify your answer by describing the behaviour of $\text{isLeaf}$, $\text{value}$, $\text{fstsubtree}$ and $\text{sndsubtree}$: for example, the reduction $\text{isLeaf} \text{(Leaf)} \Rightarrow \text{true}$ describes part of the behaviour of $\text{isLeaf}$.  

Consider the function $\text{treeadd}$ defined inductively on the structure of binary trees by

$$\text{treeadd} \ (m, \text{Leaf}) = \text{Leaf}$$
$$\text{treeadd} \ (m, \text{Br} (n, B_1, B_2)) = \text{Br} \ (m + n, \text{treeadd} \ (m, B_1), \text{treeadd} \ (m, B_2))$$

Give and justify a $\lambda$-term which encodes $\text{treeadd}$, using the $\lambda$-term $Y \equiv \lambda f. (\lambda x. f(xx))(\lambda x. f(xx))$.  

Give the $\lambda$-term for the infinite binary tree whose branch nodes consist of zeros at even depths and ones at odd depths, as pictured below:

```
0
/  \
 1   1
/ \ / \ \
0 0 0 0
/ / / / \
1 1 1 1
```

[6 marks]