1998 Paper 5 Question 12

Semantics of Programming Languages

An abstract machine for evaluating closed terms of the untyped lambda calculus has configurations which are non-empty lists of closed terms. Its transitions are of two forms:

($\overrightarrow{\text{app}}$) $\quad (M_1\, M_2) :: L \to M_1 :: M_2 :: L$

($\overrightarrow{\text{abs}}$) $\quad \lambda\, x\, (M_1) :: M_2 :: L \to M_1[M_2/x] :: L$

where :: denotes list concatenation and $M_1[M_2/x]$ denotes the result of substituting M_2 for all free occurrences of the variable x in M_1. Let $\Downarrow$ be the binary relation between closed terms inductively defined by the following axioms and rules:

($\Downarrow_{\text{abs}}$) $\quad \lambda\, x\, (M) \Downarrow \lambda\, x\, (M)$

($\Downarrow_{\text{app}}$) $$\frac{M_1 \Downarrow \lambda\, x\, (M_2) \quad M_2[M_3/x] \Downarrow \lambda\, x\, (M_4)}{M_1\, M_3 \Downarrow \lambda\, x\, (M_4)}.$$

(*a*) Prove by Rule Induction that if $M_1 \Downarrow \lambda\, x\, (M_2)$ holds, then so does $M_1 :: L \to^* \lambda\, x\, M_2 :: L$, where $\to^*$ denotes the reflexive-transitive closure of the transition relation $\to$. [5 marks]

(*b*) Prove by Mathematical Induction on n that if $(\ldots((M[M_0/x]\, M_1)M_2)\ldots)M_n \Downarrow \lambda\, x\, (M')$, then $(\ldots((((\lambda\, x\, (M))M_0)M_1)M_2)\ldots)M_n \Downarrow \lambda\, x\, (M')$. [5 marks]

(*c*) Given a configuration $M :: L$, let $M@L$ denote the closed term defined by induction on the length of the list L by: $M@\mathbf{nil} \stackrel{\text{def}}{=} M$ and $M@(M' :: L) \stackrel{\text{def}}{=} (M\, M')@L$. Using (*b*), show by case analysis for $\to$ that if $M_1 :: L_1 \to M_2 :: L_2$ and $M_2@L_2 \Downarrow \lambda\, x\, (M')$ hold, then so does $M_1@L_1 \Downarrow \lambda\, x\, (M')$. [5 marks]

(*d*) Deduce from (*a*) and (*c*) that $M_1 \Downarrow \lambda\, x\, (M_2)$ holds if and only if $M_1 :: \mathbf{nil} \to^* \lambda\, x\, (M_2) :: \mathbf{nil}$ does. [5 marks]