1998 Paper 10 Question 4

Continuous Mathematics

Show that for all families of functions which are "self-Fourier" (i.e. equivalent in functional form to their own Fourier transforms), closure of a family under multiplication entails also their closure under convolution, and *vice versa.*

[Hint: closure of a set of functions under an operation means that applying that operation to any member of the set creates a function which is also a member of the set.] [10 marks]

A periodic square wave, which alternates between the constants $+\pi/4$ and $-\pi/4$ with period 2π has the following Fourier series, using all positive odd integers n:

$$f(x) = \sum_{\text{odd } n=1}^{\infty} \frac{1}{n}\sin(nx)$$

Derive from this the Fourier series for a periodic triangular wave, which ramps up and down with slopes $+\pi/4$ and $-\pi/4$ and with period 2π. [3 marks]

Any real-valued function $f(x)$ can be represented as the sum of one function $f_e(x)$ that has even symmetry (it is unchanged after a right–left flip around $x = 0$) so that $f_e(x) = f_e(-x)$, plus one function $f_o(x)$ that has odd symmetry, so that $f_o(x) = -f_o(-x)$. Such a decomposition of any function $f(x)$ into $f_e(x) + f_o(x)$ is illustrated by

$$f_e(x) = \frac{1}{2}f(x) + \frac{1}{2}f(-x)$$

$$f_o(x) = \frac{1}{2}f(x) - \frac{1}{2}f(-x)$$

Use this type of decomposition to explain why the Fourier transform of any real-valued function has *Hermitian symmetry*: its real-part has even symmetry, and its imaginary-part has odd symmetry. Comment on how this redundancy can be exploited to simplify computation of Fourier transforms of real-valued, as opposed to complex-valued, data. [7 marks]