1998 Paper 10 Question 10

Mathematics for Computation Theory

Let $S = \{a, b\}$ be an alphabet of two characters, totally ordered by specifying that $a < b$. Let $\Sigma = S^*$ be the set of all strings over S, and $\Sigma_n = \{w \in \Sigma \mid \ell(w) = n\}$ be the subset consisting of strings of length n. For $w \in \Sigma$ with length at least n, write w_n for its initial substring of length n.

For $n \geqslant 1$ define inductively the *lexicographic order* $\sqsubseteq_n$ on $S^{(n+1)} \equiv S^{(n)} \times S$, showing that the order in each $S^{(n)}$ is *total.* [8 marks]

Defining, as usual, for s and t in $S^{(n)}$

$$s \sqsubset_n t \text{ iff } s \sqsubseteq_n t \text{ and } s \neq t$$

the *lexicographic order* $\sqsubseteq$ on Σ (often known as the *dictionary order on strings*) can be defined as follows:

$$\begin{aligned} u \sqsubseteq v \text{ iff} \quad & u_n \sqsubset_n v_n \ \ \text{(regarding } u_n \text{ and } v_n \text{ as elements of } S^{(n)}\text{)} \\ \text{or} \quad & (u_n = v_n \text{ and } \ell(u) \leqslant \ell(v)) \end{aligned}$$

where n is the shorter of the lengths of u and v. With this definition $(\Sigma, \sqsubseteq)$ is a totally ordered set.

Consider the following subsets of Σ:

$$A = \{a^n b^n \mid n \in \mathbb{N}\}$$
$$B = \{b^m a^n \mid m, n \in \mathbb{N}\}$$

For each of A, B state, giving reasons, whether it is

(a) a regular language over S

(b) a set well-ordered by $\sqsubseteq$

[12 marks]