Foundations of Functional Programming

(a) Give the definition of a head-normal form and head reduction of a $\lambda$-term. Argue that every normal form is a head-normal form. [4 marks]

(b) Let $Y_M \equiv \lambda f. W W M$, where $W \equiv \lambda x. \lambda z. f(x x z)$ and $M$ is an arbitrary term. Give a head-normal form and normal form for the following $\lambda$-terms, or indicate why they do not exist:

(i) $Y_M$

(ii) $Y_M(K I)$, where $K \equiv \lambda x. \lambda y. x$ and $I \equiv \lambda x. x$

(iii) $Y_M(K)$, where $K$ is as above

[You may assume that head reduction always terminates when a head-normal form exists.] [6 marks]

(c) A $\lambda$-term is solvable if there exist variables $x_1, \ldots, x_n$ and $\lambda$-terms $N_1, \ldots, N_m$ for $n, m \geq 0$ such that $(\lambda x_1 \ldots \lambda x_n. M) N_1 \ldots N_m = I$.

Show that every head-normal form is solvable. [4 marks]

For each term in (b), prove that it is solvable or that it is unsolvable. [6 marks]