1997 Paper 1 Question 7

Discrete Mathematics

Let us say that a finite partial order $(A, \sqsubseteq)$ is *tree-like* if, for every $a \in A$, the set (of its predecessors) $\{x \in A \mid x \sqsubseteq a \wedge x \neq a\}$ either is empty or has a unique maximal element. Equivalently, pictorially, this holds when the Hasse diagram of A consists of one or more trees.

State which of the following relations on the integers $\{1, 2, \ldots, 10\}$ are tree-like partial orders and give a one-sentence justification.

(a) R where $xRy \Leftrightarrow x = y$

(b) R where $xRy \Leftrightarrow x \leqslant y$ (here $\leqslant$ is the usual ordering on integers)

(c) R where $xRy \Leftrightarrow x$ divides-exactly-into y

(d) R where $xRy \Leftrightarrow x = y$ or x is the greatest prime factor of y

[8 marks]

To count the number $C(n)$ of tree-like partial orders of n elements, assume $A = \{1, 2, \ldots, n\}$ and then place each element i in turn into a Hasse diagram starting from 1 and such that no later element $j > i$ is placed such that $j \sqsubseteq i$.

Show that, provided $n > 1$, we have $C(n) = f(n, C(n-1))$ and give the function $f(n, m)$. Provide a base case and thereby solve the recurrence for $C(n)$. [12 marks]