1996 Paper 6 Question 12

Semantics

If D is a complete partial order and $C \subseteq D$, then we say that C is a *closed* subset of D if it satisfies the following two conditions:

- For all $x \in D$ and $y \in C$, if $x \sqsubseteq y$ then $x \in C$.
- If $x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \ldots$ is an ω-chain such that $x_n \in C$ for all n, then $(\bigsqcup_{n \in \omega} x_n) \in C$.

Prove the following statements:

(a) If C_i is a closed subset of D for all $i \in I$, then $\bigcap_{i \in I} C_i$ is a closed subset of D. [4 marks]

(b) If $C_1, C_2, \ldots, C_N$ are closed subsets of D, then $\bigcup_{1 \leqslant i \leqslant N} C_i$ is a closed subset of D. [7 marks]

(c) If $x \in D$, then $\downarrow x \stackrel{\text{def}}{=} \{y \in D \mid y \sqsubseteq x\}$ is a closed subset of D. [4 marks]

(d) If $f : E \to D$ is a continuous function, then for all closed subsets C of D, $f^{-1}(C)$ is a closed subset of E. (Where $f^{-1}(C) \stackrel{\text{def}}{=} \{x \in E \mid f(x) \in C\}$.) [5 marks]