Foundations of Functional Programming

Consider binary trees that are either empty (written $\text{Lf}$) or have the form $\text{Br } x_1 t_1 t_2$ where $t_1$ and $t_2$ are themselves binary trees. Give an encoding of binary trees in the $\lambda$-calculus, including functions $\text{isLeaf}$, $\text{label}$, $\text{left}$ and $\text{right}$ satisfying

$$
\text{isLeaf Lf} \rightarrow \text{true} \\
\text{isLeaf (Br } x_1 t_1 t_2) \rightarrow \text{false} \\
\text{label (Br } x_1 t_1 t_2) \rightarrow x \\
\text{left (Br } x_1 t_1 t_2) \rightarrow t_1 \\
\text{right (Br } x_1 t_1 t_2) \rightarrow t_2
$$

If you use encodings of other data structures, state the properties assumed.

Consider the ML functions $f$ and $g$ defined to satisfy

$$
\begin{align*}
    f([], ys) &= ys \\
    f(x :: xs, ys) &= f(xs, x :: ys) \\
    g([], ys) &= ys \\
    g(x :: xs, ys) &= x :: g(xs, ys)
\end{align*}
$$

Using list induction, prove $f(f(xs, []), []) = xs$. [14 marks]  

[Hint: generalize this formula, making use of $g$. You may assume the equation $g(xs, []) = xs$.]