1995 Paper 12 Question 14

Numerical Analysis II

A cubic spline $\phi(x)$ is defined over $[a,b]$ with knots x_1, x_2, $\ldots$ x_n such that $a < x_1$, $x_n < b$. The spline takes the values y_1, y_2, $\ldots$ y_n at the knots. What continuity conditions are usually imposed on the cubic spline at each knot? [2 marks]

If $d_j = x_{j+1} - x_j$ and $\mu_j = \phi''(x_j)$, the spline has the following formula for $x \in [x_j, x_{j+1}]$

$$\begin{aligned}\phi(x) = {} & \frac{(x-x_j)y_{j+1} + (x_{j+1}-x)y_j}{d_j} \\ & - \frac{(x-x_j)(x_{j+1}-x)\{(d_j + x_{j+1} - x)\mu_j + (d_j + x - x_j)\mu_{j+1}\}}{6d_j}.\end{aligned}$$

By differentiating this formula:

(a) find formulae for $\phi'(x_j)$ and $\phi'(x_{j+1})$ for $x \in [x_j, x_{j+1}]$ [4 marks]

(b) verify that $\phi''(x_j) = \mu_j$, $\phi''(x_{j+1}) = \mu_{j+1}$ [4 marks]

(c) deduce the equation which expresses the continuity condition on $\phi'(x)$ at x_j [6 marks]

If the equations derived in part (c) are solved as a simultaneous system, what are the unknowns? If the end conditions specify the spline to be linear in $[a, x_1]$ and $[x_n, b]$ how does this simplify the calculation? State the most important properties of the resulting equations. [4 marks]