Numerical Analysis II

Explain the terms Riemann integral and Riemann sum. [3 marks]

Let \( R \) be a quadrature rule that integrates constants exactly. If a function \( f \) is bounded and Riemann-integrable over the interval \([a, b]\) then prove that

\[
\lim_{n \to \infty} (n \times R) f = \int_{a}^{b} f(x) dx.
\]

[6 marks]

Consider two quadrature rules for the interval \([-\lambda, \lambda]\):

\[
Sf = \frac{\lambda}{3} \{f(-\lambda) + 4f(0) + f(\lambda)\} - \frac{\lambda^5}{90} f^{(4)}(\xi)
\]

\[
Tf = \lambda \{f(-\lambda) + f(\lambda)\} - \frac{2}{3} \lambda^3 f''(\zeta)
\]

If \( S \) were used in the composite form \((n \times S)f\), what order of convergence would you expect? [2 marks]

Suppose the rule

\[
\frac{1}{3} \{F(-1, -1) + 4F(-1, 0) + F(-1, 1) \\
+ F(1, -1) + 4F(1, 0) + F(1, 1)\}
\]

is applied to

\[
\int_{-1}^{1} \int_{-1}^{1} F(x, y) \, dx \, dy.
\]

Describe the 2-variable polynomials that are integrated exactly by this rule. [6 marks]

Why is the product form of \( Sf \) unsuitable for integrating over a hypercube in 20 dimensions? Name a better method for 20 dimensions on a sequential machine, given that high accuracy is not required. [3 marks]