Foundations of Functional Programming

Describe precisely the meaning and main properties of the equality \( M = N \), where \( M \) and \( N \) are terms of the \( \lambda \)-calculus. [5 marks]

In the following, consider an encoding of lists \([a_1, a_2, \ldots, a_m]\) as the \( \lambda \)-term

\[
\lambda f x. f a_1 (f a_2 \ldots (f a_m x) \ldots).
\]

Answers should include a brief justification. You may assume \( \lambda \)-encodings of the booleans and ordered pairs.

Define the \( \lambda \)-term \texttt{cons} such that

\[
\texttt{cons} [a_1, \ldots, a_m] = [a, a_1, \ldots, a_m]
\]

[2 marks]

Define the \( \lambda \)-term \texttt{null} such that

\[
\texttt{null} [a_1, \ldots, a_m] = \begin{cases} 
\text{true} & \text{(if } m = 0) \\
\text{false} & \text{(if } m > 0) 
\end{cases}
\]

[3 marks]

Define the \( \lambda \)-term \texttt{append} such that

\[
\texttt{append} [a_1, \ldots, a_m][b_1, \ldots, b_n] = [a_1, \ldots, a_m, b_1, \ldots, b_n]
\]

[3 marks]

Define the \( \lambda \)-terms \texttt{hd} and \texttt{tl} such that, if \( m > 0 \),

\[
\texttt{hd} [a_1, \ldots, a_m] = a_1 \\
\texttt{tl} [a_1, \ldots, a_m] = [a_2, \ldots, a_m]
\]

[7 marks]