1993 Paper 7 Question 8

Numerical Analysis II

If $\mathbf{B}$ is a real symmetric $n \times n$ matrix such that $\bar{\mathbf{z}}^T\mathbf{B}\mathbf{z} \geqslant 0$ for any complex vector $\mathbf{z}$, prove that any eigenvalue λ of $\mathbf{B}$ is such that $\lambda \geqslant 0$. Hence prove that the eigenvalues of $\mathbf{A}^T\mathbf{A}$, where $\mathbf{A}$ is any real square matrix, are real and non-negative. [3 marks]

Let $\mathbf{P}$, $\mathbf{Q}$ be real $n \times n$ matrices and let $\|\mathbf{P}\|_2^2$ denote the maximum eigenvalue of $\mathbf{P}^T\mathbf{P}$. State Schwarz's inequality for $\|\mathbf{PQ}\|_2$. Explain how this is modified if $\mathbf{Q}$ is replaced by a vector of n elements. [3 marks]

Derive the *condition number* K for solution of the equations $\mathbf{Ax} = \mathbf{b}$. Hint: start by setting $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$ where $\hat{\mathbf{x}}$ is an approximate solution. [5 marks]

Describe the *singular value decomposition*

$$\mathbf{A} = \mathbf{UWV}^T$$

and explain how you would use it to solve the n equations $\mathbf{Ax} = \mathbf{b}$ when $\mathbf{W}$ has rank n. [5 marks]

How may the singular value decomposition help in solving the equations $\mathbf{Ax} = \mathbf{b}$ when $\mathbf{A}$ has rank $< n$? Use the case $n = 4$, $\mathbf{W} = \mathrm{diag}\{1, 10^{-3}, 10^{-20}, 0\}$ to illustrate your answer. (You may assume that *machine epsilon* $\simeq 10^{-16}$.) [4 marks]