Numerical Analysis II

If $B$ is a real symmetric $n \times n$ matrix such that $z^T B z \geq 0$ for any complex vector $z$, prove that any eigenvalue $\lambda$ of $B$ is such that $\lambda \geq 0$. Hence prove that the eigenvalues of $A^T A$, where $A$ is any real square matrix, are real and non-negative. [3 marks]

Let $P$, $Q$ be real $n \times n$ matrices and let $\|P\|_2^2$ denote the maximum eigenvalue of $P^T P$. State Schwarz’s inequality for $\|PQ\|_2$. Explain how this is modified if $Q$ is replaced by a vector of $n$ elements. [3 marks]

Derive the condition number $K$ for solution of the equations $Ax = b$. Hint: start by setting $e = x - \hat{x}$ where $\hat{x}$ is an approximate solution. [5 marks]

Describe the singular value decomposition

$$A = UWV^T$$

and explain how you would use it to solve the $n$ equations $Ax = b$ when $W$ has rank $n$. [5 marks]

How may the singular value decomposition help in solving the equations $Ax = b$ when $A$ has rank $< n$? Use the case $n = 4$, $W = \text{diag}\{1, 10^{-3}, 10^{-20}, 0\}$ to illustrate your answer. (You may assume that machine epsilon $\approx 10^{-16}$.) [4 marks]