Foundations of Functional Programming

Describe how the $\lambda$-calculus models the operations of addition, test for zero and successor, representing the natural numbers by Church numerals. [4 marks]

The Fibonacci sequence is defined by $F_0 = 0$, $F_1 = 1$ and $F_k = F_{k-1} + F_{k-2}$ for $k \geq 2$. Present a $\lambda$-term `fib` that computes the Church numeral for $F_k$ given the Church numeral for $k$, for all $k \geq 0$. Do not use $Y$ or any other fixed point combinator. You may take as primitive the $\lambda$-calculus encodings of standard data structures. [6 marks]

Describe how to assign G"odel numbers to $\lambda$-terms and explain the notation $\lceil M \rceil$. Describe an application of these techniques. [3 marks]

Present a $\lambda$-term `iszero`, such that

$$\text{iszero} \lceil M \rceil = \begin{cases} \text{true} & \text{if } M = 0 \\ \text{false} & \text{if } M \neq 0 \end{cases}$$

or prove that no such term exists. [7 marks]