

## 1993 Paper 13 Question 9

### Introduction to Functional Programming

Consider the ML definition

```
datatype 'a tree = Lf | Br of 'a * 'a tree * 'a tree;
```

Say that a tree is *balanced* if, for each of its nodes of the form  $\text{Br}(v, t_1, t_2)$ , the number of  $\text{Br}$  nodes in  $t_1$  differs from the number of  $\text{Br}$  nodes in  $t_2$  by at most one. Express this definition succinctly as an ML function of type `'a tree -> bool`.

[2 marks]

Show how to implement functional arrays using binary trees, such that the root holds the value of subscript 1, the left subtree holds the values of even subscripts, and the right subtree holds the values of odd subscripts. Explain how the update operation works, taking subscript 6 as an example.

[5 marks]

Give an informal proof that every functional array must be represented by a balanced tree. You may assume that the array contains no gaps, i.e. the defined subscripts are  $1, \dots, n$  for some  $n > 0$ .

[7 marks]

Let the *depth* of a tree be defined by

$$\text{depth}(\text{Lf}) = 0 \quad \text{and} \quad \text{depth}(\text{Br}(v, t_1, t_2)) = 1 + \max\{\text{depth}(t_1), \text{depth}(t_2)\}.$$

State and justify precise lower and upper bounds for the number of  $\text{Br}$  nodes contained in a balanced tree of depth  $d$ , for arbitrary  $d \geq 0$ .

[6 marks]