# Randomised Algorithms

## Lecture 8: Solving a TSP Instance using Linear Programming

Thomas Sauerwald (`tms41@cam.ac.uk`)

Lent 2024

UNIVERSITY OF CAMBRIDGE

# Outline

Introduction

Examples of TSP Instances

Demonstration

# The Traveling Salesman Problem (TSP)

*Given a set of cities along with the cost of travel between them, find the cheapest route visiting all cities and returning to your starting point.*

# The Traveling Salesman Problem (TSP)

*Given a set of cities along with the cost of travel between them, find the cheapest route visiting all cities and returning to your starting point.*

Formal Definition

# The Traveling Salesman Problem (TSP)

*Given a set of cities along with the cost of travel between them, find the cheapest route visiting all cities and returning to your starting point.*

Formal Definition

- Given: A complete undirected graph $G = (V, E)$ with nonnegative integer cost $c(u, v)$ for each edge $(u, v) \in E$

# The Traveling Salesman Problem (TSP)

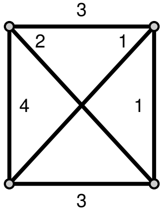
*Given a set of cities along with the cost of travel between them, find the cheapest route visiting all cities and returning to your starting point.*

Formal Definition

- Given: A complete undirected graph $G = (V, E)$ with nonnegative integer cost $c(u, v)$ for each edge $(u, v) \in E$
- Goal: Find a hamiltonian cycle of $G$ with minimum cost.

# The Traveling Salesman Problem (TSP)

*Given a set of cities along with the cost of travel between them, find the cheapest route visiting all cities and returning to your starting point.*

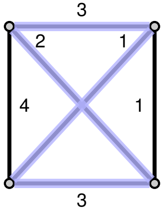



**Formal Definition**

- Given: A complete undirected graph $G = (V, E)$ with nonnegative integer cost $c(u, v)$ for each edge $(u, v) \in E$
- Goal: Find a hamiltonian cycle of $G$ with minimum cost.

# The Traveling Salesman Problem (TSP)

*Given a set of cities along with the cost of travel between them, find the cheapest route visiting all cities and returning to your starting point.*




$3 + 2 + 1 + 3 = 9$

**Formal Definition**

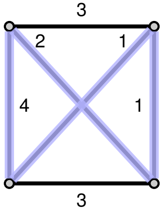
- Given: A complete undirected graph $G = (V, E)$ with nonnegative integer cost $c(u, v)$ for each edge $(u, v) \in E$
- Goal: Find a hamiltonian cycle of $G$ with minimum cost.

# The Traveling Salesman Problem (TSP)

*Given a set of cities along with the cost of travel between them, find the cheapest route visiting all cities and returning to your starting point.*

**Formal Definition**

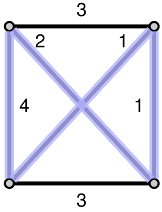
- Given: A complete undirected graph $G = (V, E)$ with nonnegative integer cost $c(u, v)$ for each edge $(u, v) \in E$
- Goal: Find a hamiltonian cycle of $G$ with minimum cost.




$2 + 4 + 1 + 1 = 8$

# The Traveling Salesman Problem (TSP)

*Given a set of cities along with the cost of travel between them, find the cheapest route visiting all cities and returning to your starting point.*




$2 + 4 + 1 + 1 = 8$

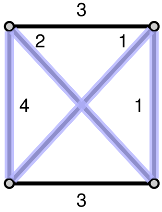
**Formal Definition**

- Given: A complete undirected graph $G = (V, E)$ with nonnegative integer cost $c(u, v)$ for each edge $(u, v) \in E$
- Goal: Find a hamiltonian cycle of $G$ with minimum cost.

Solution space consists of at most $n!$ possible tours!

# The Traveling Salesman Problem (TSP)

*Given a set of cities along with the cost of travel between them, find the cheapest route visiting all cities and returning to your starting point.*




$2 + 4 + 1 + 1 = 8$

Formal Definition

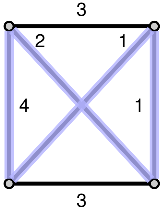
- Given: A complete undirected graph $G = (V, E)$ with nonnegative integer cost $c(u, v)$ for each edge $(u, v) \in E$
- Goal: Find a hamiltonian cycle of $G$ with minimum cost.

Solution space consists of at most $n!$ possible tours!

Actually the right number is $(n-1)!/2$

# The Traveling Salesman Problem (TSP)

*Given a set of cities along with the cost of travel between them, find the cheapest route visiting all cities and returning to your starting point.*




$2 + 4 + 1 + 1 = 8$

Formal Definition

- Given: A complete undirected graph $G = (V, E)$ with nonnegative integer cost $c(u, v)$ for each edge $(u, v) \in E$
- Goal: Find a hamiltonian cycle of $G$ with minimum cost.

Solution space consists of at most $n!$ possible tours!

Actually the right number is $(n-1)!/2$

Special Instances

# The Traveling Salesman Problem (TSP)

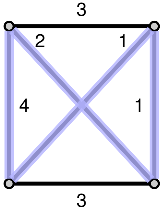
*Given a set of cities along with the cost of travel between them, find the cheapest route visiting all cities and returning to your starting point.*

Formal Definition

- Given: A complete undirected graph $G = (V, E)$ with nonnegative integer cost $c(u, v)$ for each edge $(u, v) \in E$
- Goal: Find a hamiltonian cycle of $G$ with minimum cost.

Solution space consists of at most $n!$ possible tours!

Actually the right number is $(n-1)!/2$




$2 + 4 + 1 + 1 = 8$

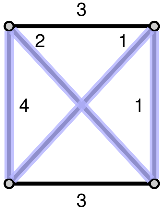
Special Instances

- Metric TSP: costs satisfy triangle inequality:

$$\forall u, v, w \in V: \qquad c(u, w) \leq c(u, v) + c(v, w).$$

# The Traveling Salesman Problem (TSP)

*Given a set of cities along with the cost of travel between them, find the cheapest route visiting all cities and returning to your starting point.*




$2 + 4 + 1 + 1 = 8$

**Formal Definition**

- Given: A complete undirected graph $G = (V, E)$ with nonnegative integer cost $c(u, v)$ for each edge $(u, v) \in E$
- Goal: Find a hamiltonian cycle of $G$ with minimum cost.

Solution space consists of at most $n!$ possible tours!

Actually the right number is $(n-1)!/2$

**Special Instances**

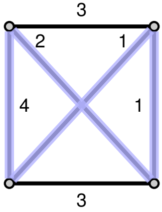
- Metric TSP: costs satisfy triangle inequality:

Even this version is NP hard (Ex. 35.2-2)

$$\forall u, v, w \in V: \qquad c(u, w) \leq c(u, v) + c(v, w).$$

# The Traveling Salesman Problem (TSP)

*Given a set of cities along with the cost of travel between them, find the cheapest route visiting all cities and returning to your starting point.*




$2 + 4 + 1 + 1 = 8$

Formal Definition

- Given: A complete undirected graph $G = (V, E)$ with nonnegative integer cost $c(u, v)$ for each edge $(u, v) \in E$
- Goal: Find a hamiltonian cycle of $G$ with minimum cost.

Solution space consists of at most $n!$ possible tours!

Actually the right number is $(n-1)!/2$

Special Instances

- Metric TSP: costs satisfy triangle inequality:

Even this version is NP hard (Ex. 35.2-2)

$$\forall u, v, w \in V : \qquad c(u, w) \leq c(u, v) + c(v, w).$$

- Euclidean TSP: cities are points in the Euclidean space, costs are equal to their (rounded) Euclidean distance

# Outline

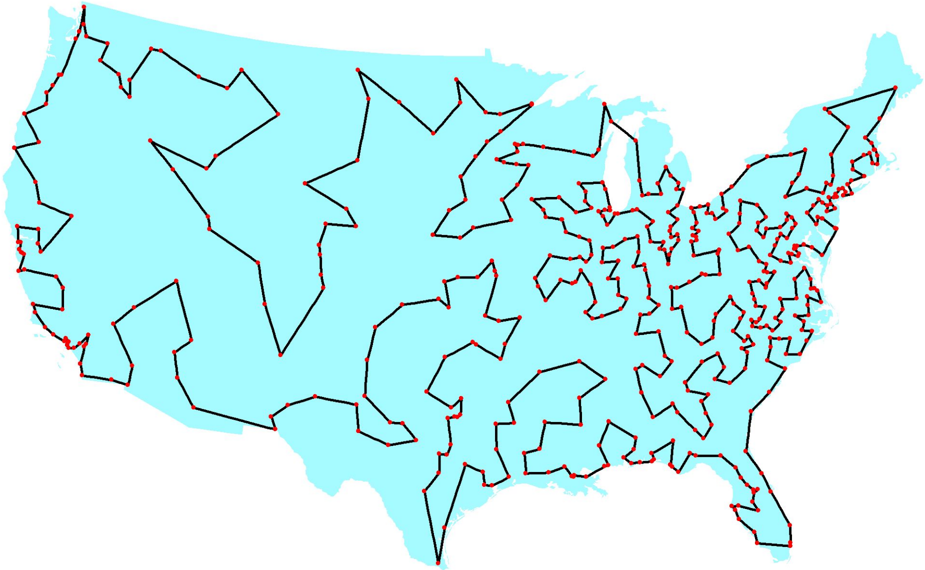
Introduction

Examples of TSP Instances

Demonstration

# 33 city contest (1964)

# 532 cities (1987 [Padberg, Rinaldi])



# 13,509 cities (1999 [Applegate, Bixby, Chavatal, Cook])

# The Original Article (1954)

SOLUTION OF A LARGE-SCALE TRAVELING-SALESMAN PROBLEM*

G. DANTZIG, R. FULKERSON, AND S. JOHNSON

*The Rand Corporation, Santa Monica, California*

(Received August 9, 1954)

It is shown that a certain tour of 49 cities, one in each of the 48 states and Washington, D. C., has the shortest road distance.

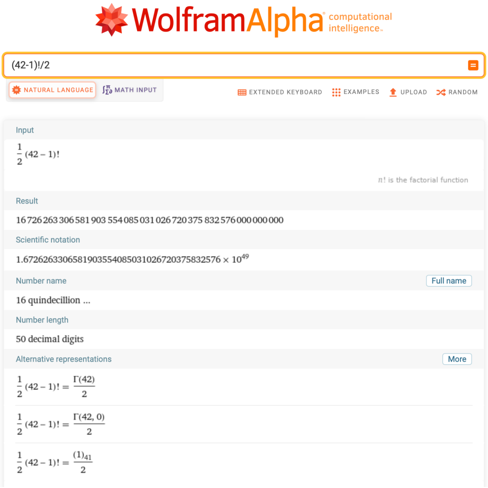
THE TRAVELING-SALESMAN PROBLEM might be described as follows: Find the shortest route (tour) for a salesman starting from a given city, visiting each of a specified group of cities, and then returning to the original point of departure. More generally, given an $n$ by $n$ symmetric matrix $D=(d_{IJ})$, where $d_{IJ}$ represents the 'distance' from $I$ to $J$, arrange the points in a cyclic order in such a way that the sum of the $d_{IJ}$ between consecutive points is minimal. Since there are only a finite number of possibilities (at most $\frac{1}{2}(n-1)!$) to consider, the problem is to devise a method of picking out the optimal arrangement which is reasonably efficient for fairly large values of $n$. Although algorithms have been devised for problems of similar nature, e.g., the optimal assignment problem,[3,7,8] little is known about the traveling-salesman problem. We do not claim that this note alters the situation very much; what we shall do is outline a way of approaching the problem that sometimes, at least, enables one to find an optimal path and prove it so. In particular, it will be shown that a certain arrangement of 49 cities, one in each of the 48 states and Washington, D. C., is best, the $d_{IJ}$ used representing road distances as taken from an atlas.

# The 42 (49) Cities

1. Manchester, N. H.
2. Montpelier, Vt.
3. Detroit, Mich.
4. Cleveland, Ohio
5. Charleston, W. Va.
6. Louisville, Ky.
7. Indianapolis, Ind.
8. Chicago, Ill.
9. Milwaukee, Wis.
10. Minneapolis, Minn.
11. Pierre, S. D.
12. Bismarck, N. D.
13. Helena, Mont.
14. Seattle, Wash.
15. Portland, Ore.
16. Boise, Idaho
17. Salt Lake City, Utah
18. Carson City, Nev.
19. Los Angeles, Calif.
20. Phoenix, Ariz.
21. Santa Fe, N. M.
22. Denver, Colo.
23. Cheyenne, Wyo.
24. Omaha, Neb.
25. Des Moines, Iowa
26. Kansas City, Mo.
27. Topeka, Kans.
28. Oklahoma City, Okla.
29. Dallas, Tex.
30. Little Rock, Ark.
31. Memphis, Tenn.
32. Jackson, Miss.
33. New Orleans, La.
34. Birmingham, Ala.
35. Atlanta, Ga.
36. Jacksonville, Fla.
37. Columbia, S. C.
38. Raleigh, N. C.
39. Richmond, Va.
40. Washington, D. C.
41. Boston, Mass.
42. Portland, Me.

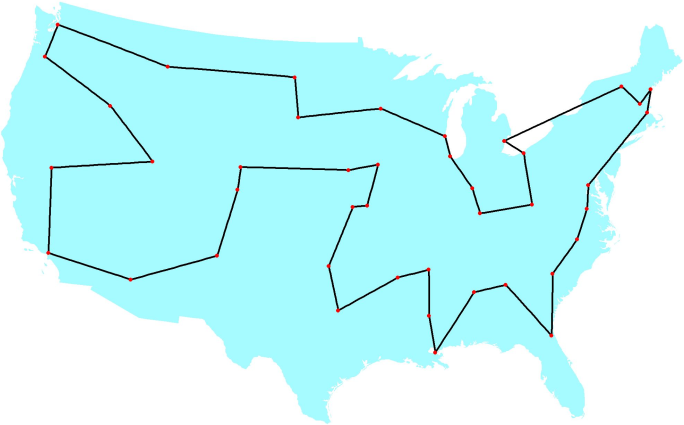
A. Baltimore, Md.
B. Wilmington, Del.
C. Philadelphia, Penn.
D. Newark, N. J.
E. New York, N. Y.
F. Hartford, Conn.
G. Providence, R. I.

# Combinatorial Explosion



# Solution of this TSP problem

Dantzig, Fulkerson and Johnson found an optimal tour through 42 cities.



http://www.math.uwaterloo.ca/tsp/history/img/dantzig_big.html

# Road Distances

TABLE I

ROAD DISTANCES BETWEEN CITIES IN ADJUSTED UNITS

The figures in the table are mileages between the two specified numbered cities, less 11, divided by 17, and rounded to the nearest integer.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **2** | 8 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| **3** | 39 | 45 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| **4** | 37 | 47 | 9 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| **5** | 50 | 49 | 21 | 15 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| **6** | 61 | 62 | 21 | 20 | 17 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| **7** | 58 | 60 | 16 | 17 | 18 | 6 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| **8** | 59 | 60 | 15 | 20 | 26 | 17 | 10 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| **9** | 62 | 66 | 20 | 25 | 31 | 22 | 15 | 5 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| **10** | 81 | 81 | 40 | 44 | 50 | 41 | 35 | 24 | 20 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| **11** | 103 | 107 | 62 | 67 | 72 | 63 | 57 | 46 | 41 | 23 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| **12** | 108 | 117 | 66 | 71 | 77 | 68 | 61 | 51 | 46 | 26 | 11 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| **13** | 145 | 149 | 104 | 108 | 114 | 106 | 99 | 88 | 84 | 63 | 49 | 40 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| **14** | 181 | 185 | 140 | 144 | 150 | 142 | 135 | 124 | 120 | 99 | 85 | 76 | 35 | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| **15** | 187 | 191 | 146 | 150 | 156 | 142 | 137 | 130 | 125 | 105 | 90 | 81 | 41 | 10 | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| **16** | 161 | 170 | 120 | 124 | 130 | 115 | 110 | 104 | 105 | 90 | 72 | 64 | 34 | 31 | 27 | | | | | | | | | | | | | | | | | | | | | | | | | | |
| **17** | 142 | 146 | 101 | 104 | 111 | 97 | 91 | 85 | 86 | 75 | 51 | 59 | 29 | 53 | 48 | 21 | | | | | | | | | | | | | | | | | | | | | | | | | |
| **18** | 174 | 178 | 133 | 138 | 143 | 129 | 123 | 117 | 118 | 107 | 83 | 84 | 54 | 46 | 35 | 26 | 31 | | | | | | | | | | | | | | | | | | | | | | | | |
| **19** | 185 | 186 | 142 | 143 | 140 | 130 | 126 | 124 | 128 | 118 | 93 | 101 | 72 | 69 | 58 | 58 | 43 | 26 | | | | | | | | | | | | | | | | | | | | | | | |
| **20** | 164 | 165 | 120 | 123 | 124 | 106 | 106 | 105 | 110 | 104 | 86 | 97 | 71 | 93 | 82 | 62 | 42 | 45 | 22 | | | | | | | | | | | | | | | | | | | | | | |
| **21** | 137 | 139 | 94 | 96 | 94 | 80 | 78 | 77 | 84 | 77 | 56 | 64 | 65 | 90 | 87 | 58 | 36 | 68 | 50 | 30 | | | | | | | | | | | | | | | | | | | | | |
| **22** | 117 | 122 | 77 | 80 | 83 | 68 | 62 | 60 | 61 | 50 | 34 | 42 | 49 | 82 | 77 | 60 | 30 | 62 | 70 | 49 | 21 | | | | | | | | | | | | | | | | | | | | |
| **23** | 114 | 118 | 73 | 78 | 84 | 69 | 63 | 57 | 59 | 48 | 28 | 36 | 43 | 77 | 72 | 45 | 27 | 59 | 69 | 55 | 27 | 5 | | | | | | | | | | | | | | | | | | | |
| **24** | 85 | 89 | 44 | 48 | 53 | 41 | 34 | 28 | 29 | 22 | 23 | 35 | 69 | 105 | 102 | 74 | 56 | 88 | 99 | 81 | 54 | 32 | 29 | | | | | | | | | | | | | | | | | | |
| **25** | 77 | 80 | 36 | 40 | 46 | 34 | 27 | 19 | 21 | 14 | 29 | 40 | 77 | 114 | 111 | 84 | 64 | 96 | 107 | 87 | 60 | 40 | 37 | 8 | | | | | | | | | | | | | | | | | |
| **26** | 87 | 89 | 44 | 46 | 46 | 30 | 28 | 29 | 32 | 27 | 36 | 47 | 78 | 116 | 112 | 84 | 66 | 98 | 95 | 75 | 47 | 36 | 39 | 12 | 11 | | | | | | | | | | | | | | | | |
| **27** | 91 | 93 | 48 | 50 | 48 | 34 | 32 | 33 | 36 | 30 | 34 | 45 | 77 | 115 | 110 | 83 | 63 | 97 | 91 | 72 | 44 | 32 | 36 | 9 | 15 | 3 | | | | | | | | | | | | | | | |
| **28** | 105 | 106 | 62 | 63 | 64 | 47 | 46 | 49 | 54 | 48 | 46 | 59 | 85 | 119 | 115 | 88 | 66 | 98 | 79 | 59 | 31 | 36 | 42 | 28 | 33 | 21 | 20 | | | | | | | | | | | | | | |
| **29** | 111 | 113 | 69 | 71 | 66 | 51 | 53 | 56 | 61 | 57 | 59 | 71 | 96 | 130 | 126 | 98 | 75 | 98 | 85 | 62 | 38 | 47 | 53 | 39 | 42 | 29 | 30 | 12 | | | | | | | | | | | | | |
| **30** | 91 | 92 | 50 | 51 | 46 | 30 | 34 | 38 | 43 | 49 | 60 | 71 | 103 | 141 | 136 | 109 | 90 | 115 | 99 | 81 | 53 | 61 | 62 | 36 | 34 | 24 | 28 | 20 | 20 | | | | | | | | | | | | |
| **31** | 83 | 85 | 42 | 43 | 38 | 22 | 26 | 32 | 36 | 51 | 63 | 75 | 106 | 142 | 140 | 112 | 93 | 126 | 108 | 88 | 60 | 64 | 66 | 39 | 36 | 27 | 31 | 28 | 28 | 8 | | | | | | | | | | | |
| **32** | 89 | 91 | 55 | 55 | 50 | 34 | 39 | 44 | 49 | 63 | 76 | 87 | 120 | 155 | 150 | 123 | 100 | 123 | 109 | 86 | 62 | 71 | 78 | 52 | 49 | 39 | 44 | 35 | 24 | 15 | 12 | | | | | | | | | | |
| **33** | 95 | 97 | 64 | 63 | 56 | 42 | 49 | 56 | 60 | 75 | 86 | 97 | 126 | 160 | 155 | 128 | 104 | 128 | 113 | 90 | 67 | 76 | 82 | 62 | 59 | 49 | 53 | 40 | 29 | 25 | 23 | 11 | | | | | | | | | |
| **34** | 74 | 81 | 44 | 43 | 35 | 23 | 30 | 39 | 44 | 62 | 78 | 89 | 121 | 159 | 155 | 127 | 108 | 136 | 124 | 101 | 75 | 79 | 81 | 54 | 50 | 42 | 46 | 43 | 39 | 23 | 14 | 14 | 21 | | | | | | | | |
| **35** | 67 | 69 | 42 | 41 | 31 | 25 | 32 | 41 | 46 | 64 | 83 | 90 | 130 | 164 | 160 | 133 | 114 | 146 | 134 | 111 | 85 | 84 | 86 | 59 | 52 | 47 | 51 | 53 | 49 | 32 | 24 | 24 | 30 | 9 | | | | | | | |
| **36** | 74 | 76 | 61 | 60 | 42 | 44 | 51 | 60 | 66 | 83 | 102 | 110 | 147 | 185 | 179 | 155 | 133 | 159 | 146 | 122 | 98 | 105 | 107 | 79 | 71 | 66 | 70 | 70 | 60 | 48 | 40 | 36 | 33 | 25 | 18 | | | | | | |
| **37** | 57 | 59 | 46 | 41 | 25 | 30 | 36 | 47 | 52 | 71 | 93 | 98 | 136 | 172 | 172 | 148 | 126 | 158 | 147 | 124 | 121 | 97 | 99 | 71 | 65 | 59 | 63 | 67 | 62 | 46 | 38 | 37 | 43 | 23 | 13 | 17 | | | | | |
| **38** | 45 | 46 | 41 | 34 | 20 | 34 | 38 | 48 | 53 | 73 | 96 | 99 | 137 | 176 | 178 | 151 | 131 | 163 | 159 | 135 | 108 | 102 | 103 | 73 | 67 | 64 | 69 | 75 | 72 | 54 | 46 | 49 | 54 | 34 | 24 | 29 | 12 | | | | |
| **39** | 35 | 37 | 35 | 26 | 18 | 34 | 36 | 46 | 51 | 70 | 93 | 97 | 134 | 171 | 176 | 151 | 129 | 161 | 163 | 139 | 118 | 102 | 101 | 71 | 65 | 65 | 70 | 84 | 78 | 58 | 50 | 56 | 62 | 41 | 32 | 38 | 21 | 9 | | | |
| **40** | 29 | 33 | 30 | 21 | 18 | 35 | 33 | 40 | 45 | 65 | 87 | 91 | 117 | 166 | 171 | 144 | 125 | 157 | 156 | 139 | 113 | 95 | 97 | 67 | 60 | 62 | 67 | 79 | 82 | 62 | 53 | 59 | 66 | 45 | 38 | 45 | 27 | 15 | 6 | | |
| **41** | 3 | 11 | 41 | 37 | 47 | 57 | 55 | 58 | 63 | 83 | 105 | 109 | 147 | 186 | 188 | 164 | 144 | 176 | 182 | 161 | 134 | 119 | 116 | 86 | 78 | 84 | 88 | 101 | 108 | 88 | 80 | 86 | 92 | 71 | 64 | 71 | 54 | 41 | 32 | 25 | |
| **42** | 5 | 12 | 55 | 41 | 53 | 64 | 61 | 61 | 66 | 84 | 111 | 113 | 150 | 186 | 192 | 166 | 147 | 180 | 188 | 167 | 140 | 124 | 119 | 90 | 87 | 90 | 94 | 107 | 114 | 77 | 86 | 92 | 98 | 80 | 74 | 77 | 60 | 48 | 38 | 32 | 6 |

# Road Distances

Hence this is an instance of the Metric TSP, but not Euclidean TSP.

TABLE I

ROAD DISTANCES BETWEEN CITIES IN ADJUSTED UNITS

The figures in the table are mileages between the two specified numbered cities, less 11, divided by 17, and rounded to the nearest integer.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 8 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 3 | 39 | 45 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 4 | 37 | 47 | 9 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 5 | 50 | 49 | 21 | 15 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 6 | 61 | 62 | 21 | 20 | 17 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 7 | 58 | 60 | 16 | 17 | 18 | 6 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 8 | 59 | 60 | 15 | 20 | 26 | 17 | 10 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 9 | 62 | 66 | 20 | 25 | 31 | 22 | 15 | 5 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 10 | 81 | 81 | 40 | 44 | 50 | 41 | 35 | 24 | 20 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 11 | 103 | 107 | 62 | 67 | 72 | 63 | 57 | 46 | 41 | 23 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 12 | 108 | 117 | 66 | 71 | 77 | 68 | 61 | 51 | 46 | 26 | 11 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 13 | 145 | 149 | 104 | 108 | 114 | 106 | 99 | 88 | 84 | 63 | 49 | 40 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 14 | 181 | 185 | 140 | 144 | 150 | 142 | 135 | 124 | 120 | 99 | 85 | 76 | 35 | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 15 | 187 | 191 | 146 | 150 | 156 | 142 | 137 | 130 | 125 | 105 | 90 | 81 | 41 | 10 | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 16 | 161 | 170 | 120 | 124 | 130 | 115 | 110 | 104 | 105 | 90 | 72 | 64 | 34 | 31 | 27 | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 17 | 142 | 146 | 101 | 104 | 111 | 97 | 91 | 85 | 86 | 75 | 51 | 59 | 29 | 53 | 48 | 21 | | | | | | | | | | | | | | | | | | | | | | | | | |
| 18 | 174 | 178 | 133 | 138 | 143 | 129 | 123 | 117 | 118 | 107 | 83 | 84 | 54 | 46 | 35 | 26 | 31 | | | | | | | | | | | | | | | | | | | | | | | | |
| 19 | 185 | 186 | 142 | 143 | 140 | 130 | 126 | 124 | 128 | 118 | 93 | 101 | 72 | 69 | 58 | 58 | 43 | 26 | | | | | | | | | | | | | | | | | | | | | | | |
| 20 | 164 | 165 | 120 | 123 | 124 | 106 | 106 | 105 | 110 | 104 | 86 | 97 | 71 | 93 | 82 | 62 | 42 | 45 | 22 | | | | | | | | | | | | | | | | | | | | | | |
| 21 | 137 | 139 | 94 | 96 | 94 | 80 | 78 | 77 | 84 | 77 | 56 | 64 | 65 | 90 | 87 | 58 | 36 | 68 | 50 | 30 | | | | | | | | | | | | | | | | | | | | | |
| 22 | 117 | 122 | 77 | 80 | 83 | 68 | 62 | 60 | 61 | 50 | 34 | 42 | 49 | 82 | 77 | 60 | 30 | 62 | 70 | 49 | 21 | | | | | | | | | | | | | | | | | | | | |
| 23 | 114 | 118 | 73 | 78 | 84 | 69 | 63 | 57 | 59 | 48 | 28 | 36 | 43 | 77 | 72 | 45 | 27 | 59 | 69 | 55 | 27 | 5 | | | | | | | | | | | | | | | | | | | |
| 24 | 85 | 89 | 44 | 48 | 53 | 41 | 34 | 28 | 29 | 22 | 23 | 35 | 69 | 105 | 102 | 74 | 56 | 88 | 99 | 81 | 54 | 32 | 29 | | | | | | | | | | | | | | | | | | |
| 25 | 77 | 80 | 36 | 40 | 46 | 34 | 27 | 19 | 21 | 14 | 29 | 40 | 77 | 114 | 111 | 84 | 64 | 96 | 107 | 87 | 60 | 40 | 37 | 8 | | | | | | | | | | | | | | | | | |
| 26 | 87 | 89 | 44 | 46 | 46 | 30 | 28 | 29 | 32 | 27 | 36 | 47 | 78 | 116 | 112 | 84 | 66 | 98 | 95 | 75 | 47 | 36 | 39 | 12 | 11 | | | | | | | | | | | | | | | | |
| 27 | 91 | 93 | 48 | 50 | 48 | 34 | 32 | 33 | 36 | 30 | 34 | 45 | 77 | 115 | 110 | 83 | 63 | 97 | 91 | 72 | 44 | 32 | 36 | 9 | 15 | 3 | | | | | | | | | | | | | | | |
| 28 | 105 | 106 | 62 | 63 | 64 | 47 | 46 | 49 | 54 | 48 | 46 | 59 | 85 | 119 | 115 | 88 | 66 | 98 | 79 | 59 | 31 | 36 | 42 | 28 | 33 | 21 | 20 | | | | | | | | | | | | | | |
| 29 | 111 | 113 | 69 | 71 | 66 | 51 | 53 | 56 | 61 | 57 | 59 | 71 | 96 | 130 | 126 | 98 | 75 | 98 | 85 | 62 | 38 | 47 | 53 | 39 | 42 | 29 | 30 | 12 | | | | | | | | | | | | | |
| 30 | 91 | 92 | 50 | 51 | 46 | 30 | 34 | 38 | 43 | 49 | 60 | 71 | 103 | 141 | 136 | 109 | 90 | 115 | 99 | 81 | 53 | 61 | 62 | 36 | 34 | 24 | 28 | 20 | 20 | | | | | | | | | | | | |
| 31 | 83 | 85 | 42 | 43 | 38 | 22 | 26 | 32 | 36 | 51 | 63 | 75 | 106 | 142 | 140 | 112 | 93 | 126 | 108 | 88 | 60 | 64 | 66 | 39 | 36 | 27 | 31 | 28 | 28 | 8 | | | | | | | | | | | |
| 32 | 89 | 91 | 55 | 55 | 50 | 34 | 39 | 44 | 49 | 63 | 76 | 87 | 120 | 155 | 150 | 123 | 100 | 123 | 109 | 86 | 62 | 71 | 78 | 52 | 49 | 39 | 44 | 35 | 24 | 15 | 12 | | | | | | | | | | |
| 33 | 95 | 97 | 64 | 63 | 56 | 42 | 49 | 56 | 60 | 75 | 86 | 97 | 126 | 160 | 155 | 128 | 104 | 128 | 113 | 90 | 67 | 76 | 82 | 62 | 59 | 49 | 53 | 40 | 29 | 25 | 23 | 11 | | | | | | | | | |
| 34 | 74 | 81 | 44 | 43 | 35 | 23 | 30 | 39 | 44 | 62 | 78 | 89 | 121 | 159 | 155 | 127 | 108 | 136 | 124 | 101 | 75 | 79 | 81 | 54 | 50 | 42 | 46 | 43 | 39 | 23 | 14 | 14 | 21 | | | | | | | | |
| 35 | 67 | 69 | 42 | 41 | 31 | 25 | 32 | 41 | 46 | 64 | 83 | 90 | 130 | 164 | 160 | 133 | 114 | 146 | 134 | 111 | 85 | 84 | 86 | 59 | 52 | 47 | 51 | 53 | 49 | 32 | 24 | 24 | 30 | 9 | | | | | | | |
| 36 | 74 | 76 | 61 | 60 | 42 | 44 | 51 | 60 | 66 | 83 | 102 | 110 | 147 | 185 | 179 | 155 | 133 | 159 | 146 | 122 | 98 | 105 | 107 | 79 | 71 | 66 | 70 | 70 | 60 | 48 | 40 | 36 | 33 | 25 | 18 | | | | | | |
| 37 | 57 | 59 | 46 | 41 | 25 | 30 | 36 | 47 | 52 | 71 | 93 | 98 | 136 | 172 | 172 | 148 | 126 | 158 | 147 | 124 | 121 | 97 | 99 | 71 | 65 | 59 | 63 | 67 | 62 | 46 | 38 | 37 | 43 | 23 | 13 | 17 | | | | | |
| 38 | 45 | 46 | 41 | 34 | 20 | 34 | 38 | 48 | 53 | 73 | 96 | 99 | 137 | 176 | 178 | 151 | 131 | 163 | 159 | 135 | 108 | 102 | 103 | 73 | 67 | 64 | 69 | 75 | 72 | 54 | 46 | 49 | 54 | 34 | 24 | 29 | 12 | | | | |
| 39 | 35 | 37 | 35 | 26 | 18 | 34 | 36 | 46 | 51 | 70 | 93 | 97 | 134 | 171 | 176 | 151 | 129 | 161 | 163 | 139 | 118 | 102 | 101 | 71 | 65 | 65 | 70 | 84 | 78 | 58 | 50 | 56 | 62 | 41 | 32 | 38 | 21 | 9 | | | |
| 40 | 29 | 33 | 30 | 21 | 18 | 35 | 33 | 40 | 45 | 65 | 87 | 91 | 117 | 166 | 171 | 144 | 125 | 157 | 156 | 139 | 113 | 95 | 97 | 67 | 60 | 62 | 67 | 79 | 82 | 62 | 53 | 59 | 66 | 45 | 38 | 45 | 27 | 15 | 6 | | |
| 41 | 3 | 11 | 41 | 37 | 47 | 57 | 55 | 58 | 63 | 83 | 105 | 109 | 147 | 186 | 188 | 164 | 144 | 176 | 182 | 161 | 134 | 119 | 116 | 86 | 78 | 84 | 88 | 101 | 108 | 88 | 80 | 86 | 92 | 71 | 64 | 71 | 54 | 41 | 32 | 25 | |
| 42 | 5 | 12 | 55 | 41 | 53 | 64 | 61 | 61 | 66 | 84 | 111 | 113 | 150 | 186 | 192 | 166 | 147 | 180 | 188 | 167 | 140 | 124 | 119 | 90 | 87 | 90 | 94 | 107 | 114 | 77 | 86 | 92 | 98 | 80 | 74 | 77 | 60 | 48 | 38 | 32 | 6 |

# Modelling TSP as a Linear Program Relaxation

Idea: Indicator variable $x(i,j)$, $i > j$, which is one if the tour includes edge $\{i,j\}$ (in either direction)

# Modelling TSP as a Linear Program Relaxation

Idea: Indicator variable $x(i,j)$, $i > j$, which is one if the tour includes edge $\{i,j\}$ (in either direction)

minimize $\quad \sum_{i=1}^{42} \sum_{j=1}^{i-1} c(i,j)x(i,j)$

subject to

$$\sum_{j<i} x(i,j) + \sum_{j>i} x(j,i) = 2 \qquad \text{for each } 1 \leq i \leq 42$$

$$0 \leq x(i,j) \leq 1 \qquad \text{for each } 1 \leq j < i \leq 42$$

# Modelling TSP as a Linear Program Relaxation

Idea: Indicator variable $x(i,j)$, $i > j$, which is one if the tour includes edge $\{i,j\}$ (in either direction)

minimize $\sum_{i=1}^{42} \sum_{j=1}^{i-1} c(i,j) x(i,j)$

subject to

$$\sum_{j<i} x(i,j) + \sum_{j>i} x(j,i) = 2 \qquad \text{for each } 1 \leq i \leq 42$$

$$0 \leq x(i,j) \leq 1 \qquad \text{for each } 1 \leq j < i \leq 42$$

Constraints $x(i,j) \in \{0,1\}$ are not allowed in a LP!

# Modelling TSP as a Linear Program Relaxation

Idea: Indicator variable $x(i,j)$, $i > j$, which is one if the tour includes edge $\{i,j\}$ (in either direction)

minimize $\sum_{i=1}^{42} \sum_{j=1}^{i-1} c(i,j)x(i,j)$

subject to

$$\sum_{j<i} x(i,j) + \sum_{j>i} x(j,i) = 2 \qquad \text{for each } 1 \leq i \leq 42$$

$$0 \leq x(i,j) \leq 1 \qquad \text{for each } 1 \leq j < i \leq 42$$

Constraints $x(i,j) \in \{0,1\}$ are not allowed in a LP!

**Branch & Bound to solve an Integer Program**:

# Modelling TSP as a Linear Program Relaxation

Idea: Indicator variable $x(i,j)$, $i > j$, which is one if the tour includes edge $\{i,j\}$ (in either direction)

minimize $\quad \sum_{i=1}^{42} \sum_{j=1}^{i-1} c(i,j) x(i,j)$

subject to

$$\sum_{j<i} x(i,j) + \sum_{j>i} x(j,i) = 2 \qquad \text{for each } 1 \leq i \leq 42$$

$$0 \leq x(i,j) \leq 1 \qquad \text{for each } 1 \leq j < i \leq 42$$

Constraints $x(i,j) \in \{0,1\}$ are not allowed in a LP!

**Branch & Bound to solve an Integer Program**:

- As long as solution of LP has fractional $x(i,j) \in (0,1)$:

# Modelling TSP as a Linear Program Relaxation

Idea: Indicator variable $x(i, j)$, $i > j$, which is one if the tour includes edge $\{i, j\}$ (in either direction)

minimize $\sum_{i=1}^{42} \sum_{j=1}^{i-1} c(i, j)x(i, j)$

subject to

$$\sum_{j<i} x(i, j) + \sum_{j>i} x(j, i) = 2 \qquad \text{for each } 1 \leq i \leq 42$$

$$0 \leq x(i, j) \leq 1 \qquad \text{for each } 1 \leq j < i \leq 42$$

Constraints $x(i, j) \in \{0, 1\}$ are not allowed in a LP!

**Branch & Bound to solve an Integer Program**:

- As long as solution of LP has fractional $x(i, j) \in (0, 1)$:
  - Add $x(i, j) = 0$ to the LP, solve it and recurse
  - Add $x(i, j) = 1$ to the LP, solve it and recurse
  - Return best of these two solutions

# Modelling TSP as a Linear Program Relaxation

Idea: Indicator variable $x(i,j)$, $i > j$, which is one if the tour includes edge $\{i,j\}$ (in either direction)

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{42} \sum_{j=1}^{i-1} c(i,j) x(i,j) \\ \text{subject to} & \\ & \sum_{j<i} x(i,j) + \sum_{j>i} x(j,i) = 2 \quad \text{for each } 1 \leq i \leq 42 \\ & 0 \leq x(i,j) \leq 1 \quad \text{for each } 1 \leq j < i \leq 42 \end{array}$$

Constraints $x(i,j) \in \{0,1\}$ are not allowed in a LP!

**Branch & Bound to solve an Integer Program**:
- As long as solution of LP has fractional $x(i,j) \in (0,1)$:
  - Add $x(i,j) = 0$ to the LP, solve it and recurse
  - Add $x(i,j) = 1$ to the LP, solve it and recurse
  - Return best of these two solutions
- If solution of LP integral, return objective value

# Modelling TSP as a Linear Program Relaxation

Idea: Indicator variable $x(i,j)$, $i > j$, which is one if the tour includes edge $\{i,j\}$ (in either direction)

minimize $\sum_{i=1}^{42} \sum_{j=1}^{i-1} c(i,j)x(i,j)$

subject to

$$\sum_{j<i} x(i,j) + \sum_{j>i} x(j,i) = 2 \qquad \text{for each } 1 \leq i \leq 42$$

$$0 \leq x(i,j) \leq 1 \qquad \text{for each } 1 \leq j < i \leq 42$$

Constraints $x(i,j) \in \{0,1\}$ are not allowed in a LP!

**Branch & Bound to solve an Integer Program**:

- As long as solution of LP has fractional $x(i,j) \in (0,1)$:
  - Add $x(i,j) = 0$ to the LP, solve it and recurse
  - Add $x(i,j) = 1$ to the LP, solve it and recurse
  - Return best of these two solutions
- If solution of LP integral, return objective value

**Bound-Step**: If the best known integral solution so far is better than the solution of a LP, no need to explore branch further!

# Outline

Introduction

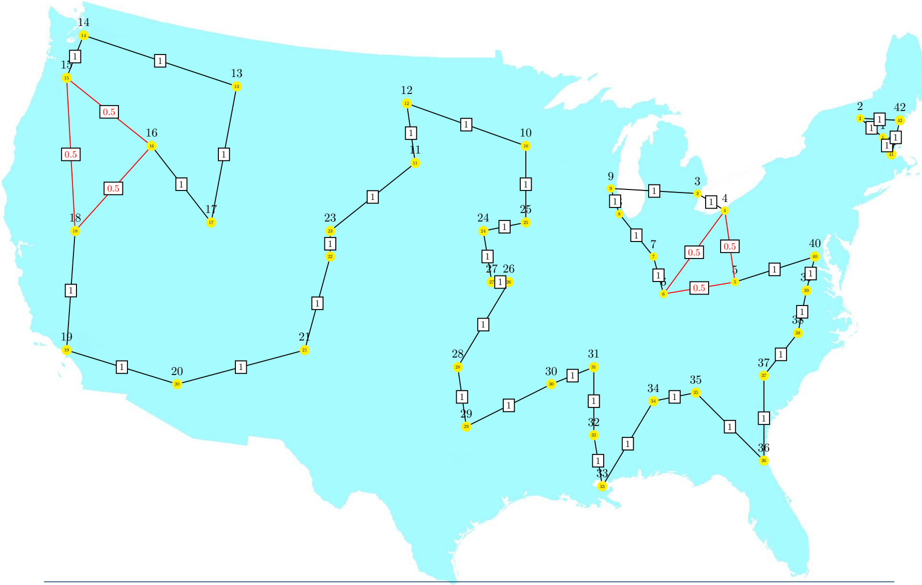
Examples of TSP Instances

Demonstration

In the following, there are a few different runs of the demo.

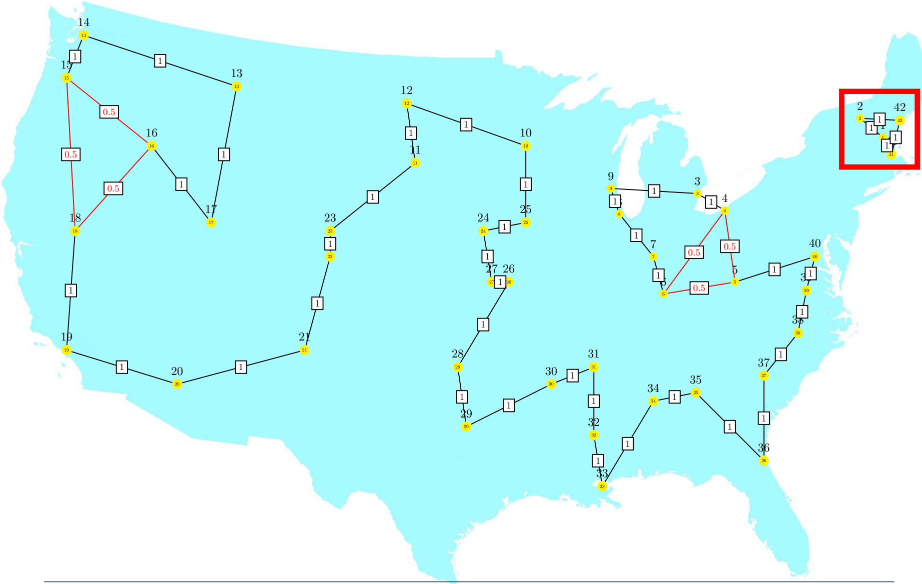
## Iteration 1:

Objective value: $-641.000000$, 861 variables, 945 constraints, 1809 iterations



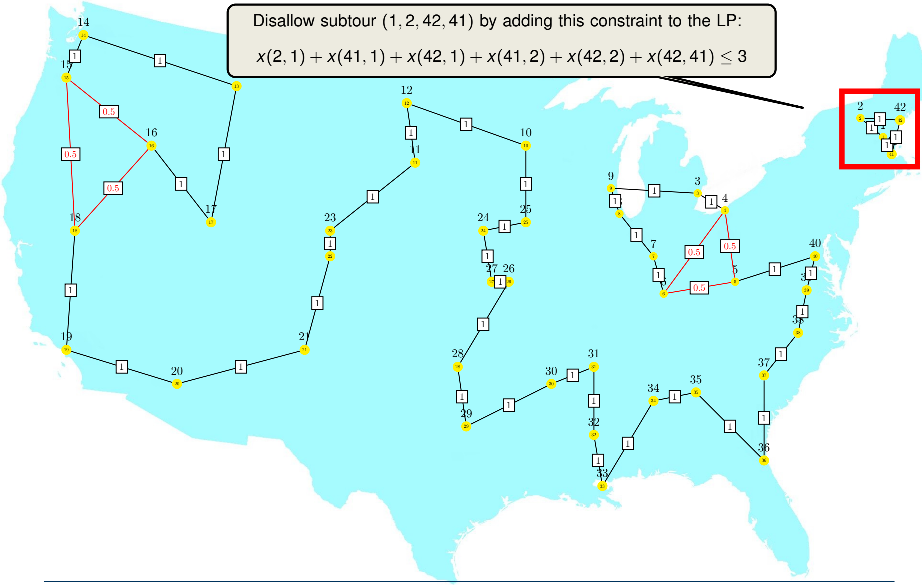
## Iteration 1: Eliminate Subtour $1, 2, 41, 42$

Objective value: $-641.000000$, 861 variables, 945 constraints, 1809 iterations



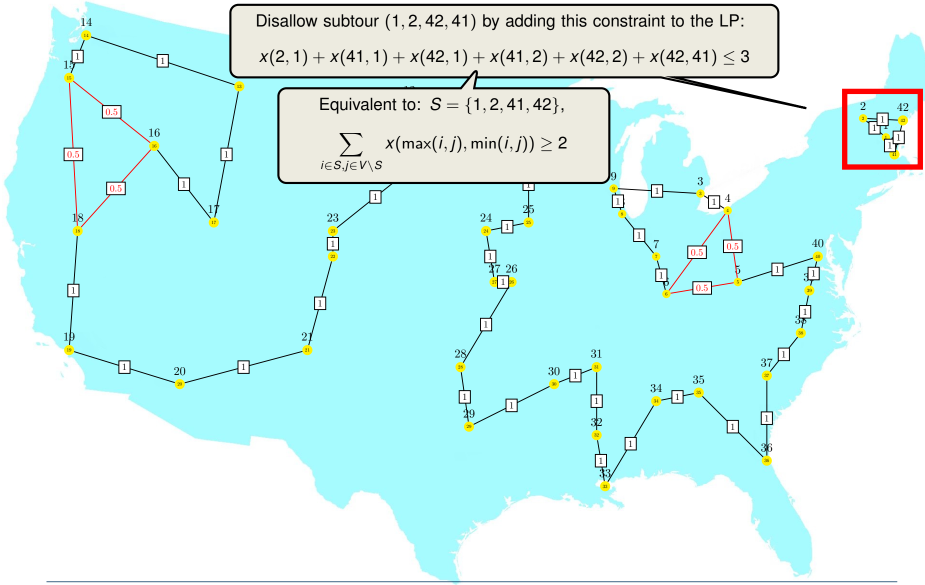
## Iteration 1: Eliminate Subtour $1, 2, 41, 42$

Objective value: $-641.000000$, 861 variables, 945 constraints, 1809 iterations



# Iteration 1: Eliminate Subtour $1, 2, 41, 42$

Objective value: $-641.000000$, 861 variables, 945 constraints, 1809 iterations



Disallow subtour $(1, 2, 42, 41)$ by adding this constraint to the LP:

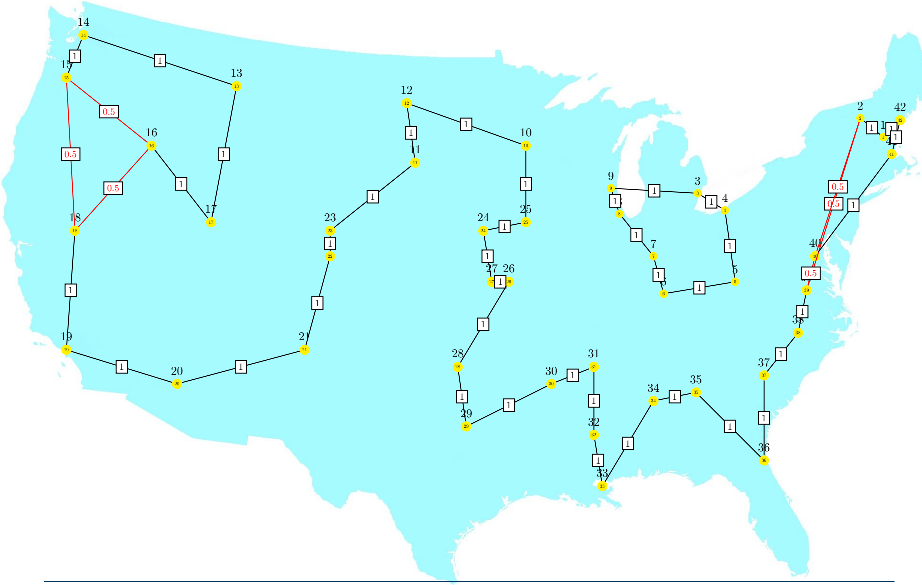
$$x(2,1) + x(41,1) + x(42,1) + x(41,2) + x(42,2) + x(42,41) \leq 3$$

Equivalent to: $S = \{1, 2, 41, 42\}$,

$$\sum_{i \in S, j \in V \setminus S} x(\max(i,j), \min(i,j)) \geq 2$$

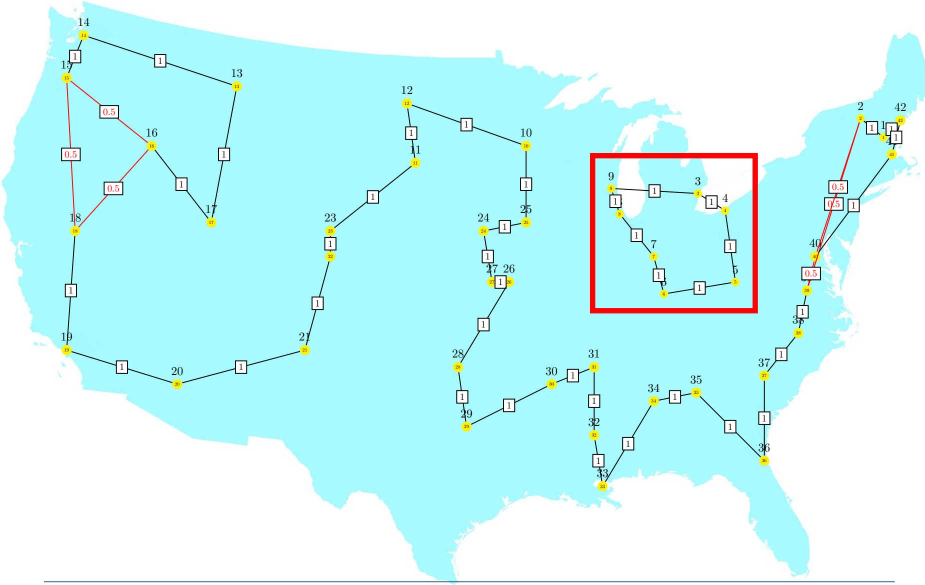
## Iteration 2:

Objective value: $-676.000000$, 861 variables, 946 constraints, 1802 iterations



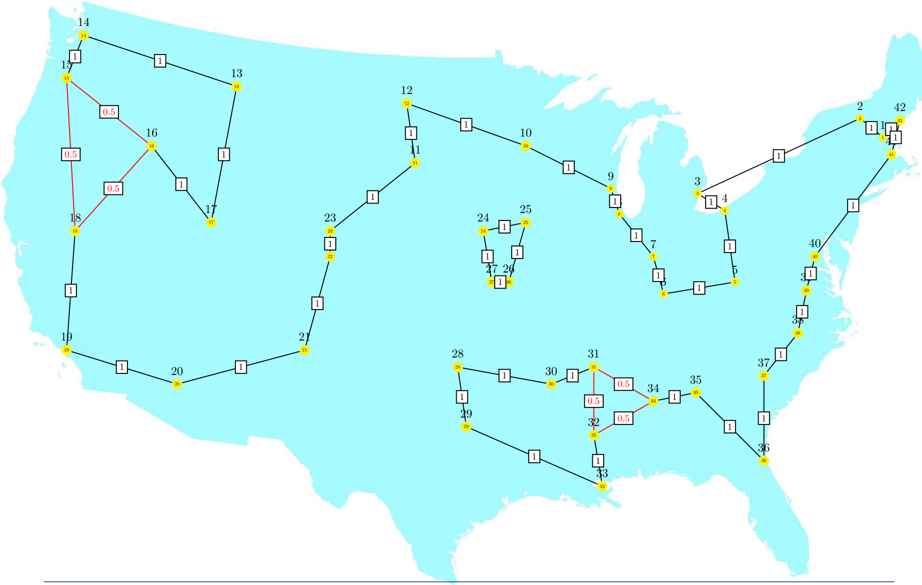
# Iteration 2: Eliminate Subtour $3 - 9$

Objective value: $-676.000000$, 861 variables, 946 constraints, 1802 iterations

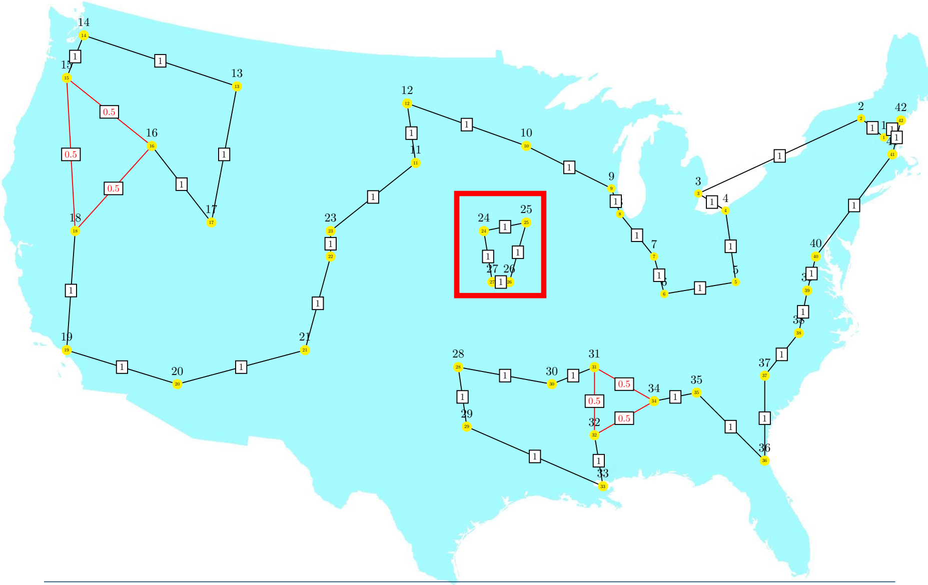


## Iteration 3:

Objective value: $-681.000000$, 861 variables, 947 constraints, 1984 iterations

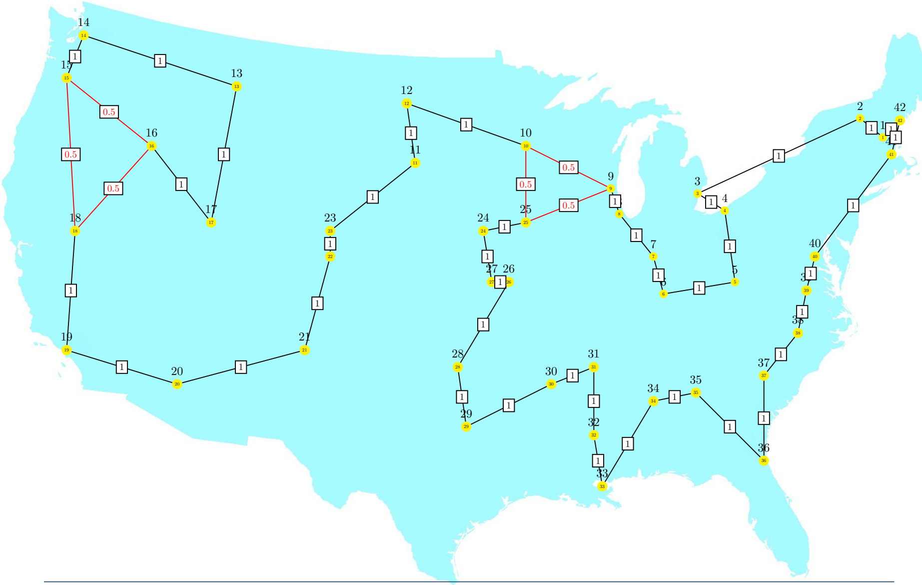


Objective value: $-681.000000$, 861 variables, 947 constraints, 1984 iterations



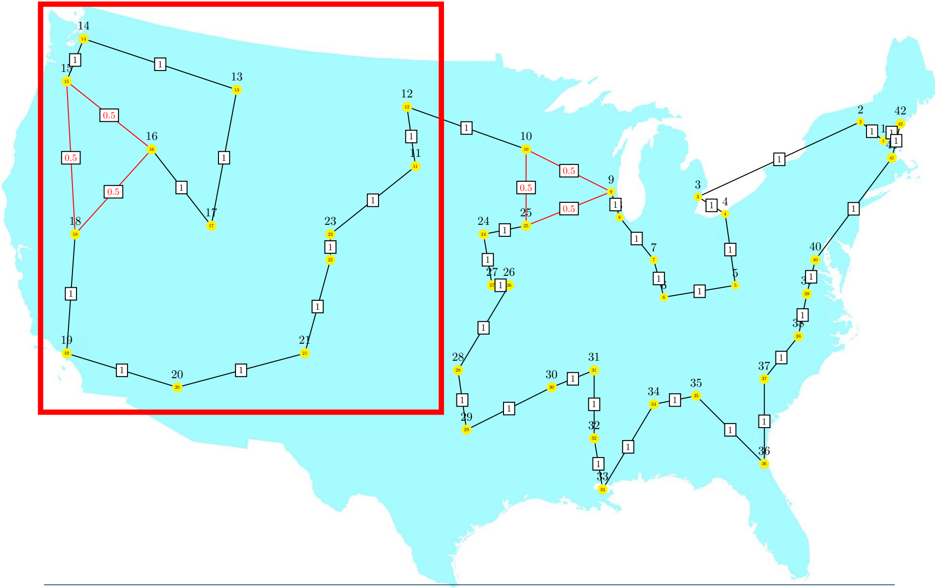
# Iteration 4:

Objective value: $-682.500000$, 861 variables, 948 constraints, 1492 iterations

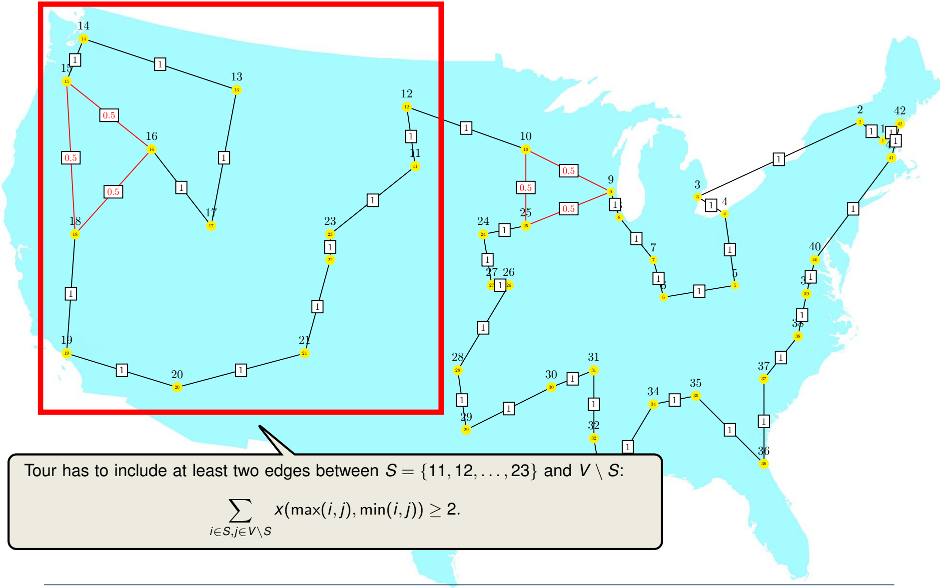


## Iteration 4: Eliminate Cut $11 - 23$

Objective value: $-682.500000$, 861 variables, 948 constraints, 1492 iterations



Objective value: $-682.500000$, 861 variables, 948 constraints, 1492 iterations

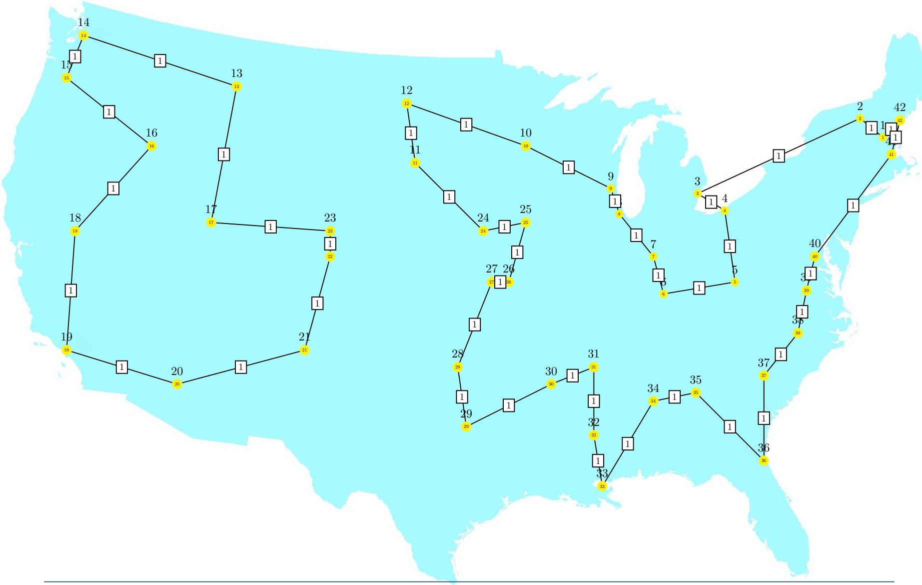



Tour has to include at least two edges between $S = \{11, 12, \ldots, 23\}$ and $V \setminus S$:

$$\sum_{i \in S, j \in V \setminus S} x(\max(i,j), \min(i,j)) \geq 2.$$

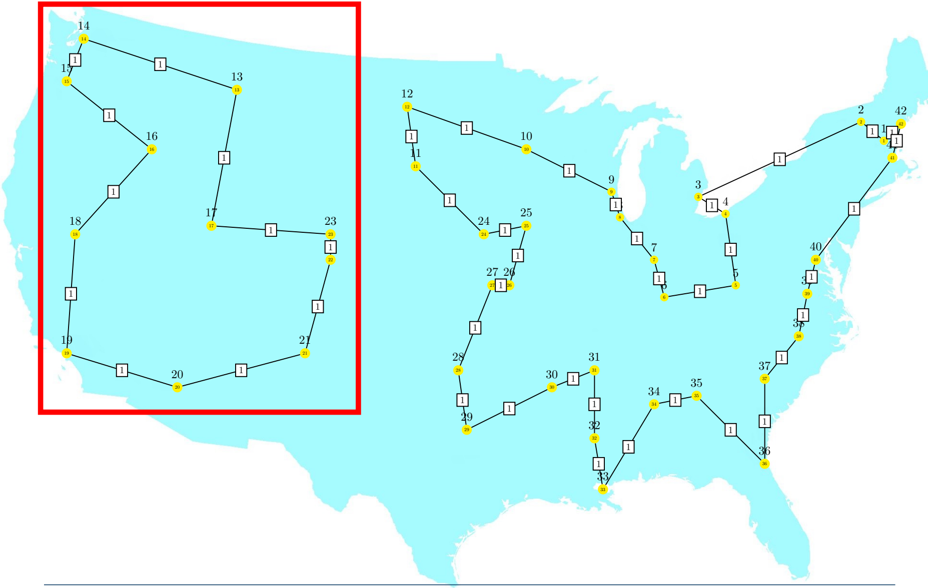
## Iteration 5:

Objective value: $-686.000000$, 861 variables, 949 constraints, 2446 iterations



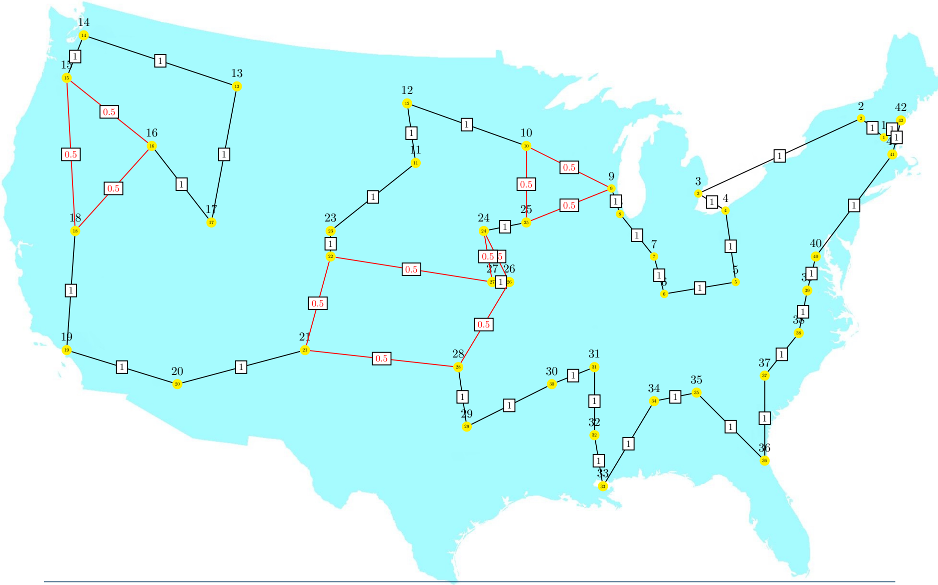
# Iteration 5: Eliminate Subtour $13 - 23$

Objective value: $-686.000000$, 861 variables, 949 constraints, 2446 iterations



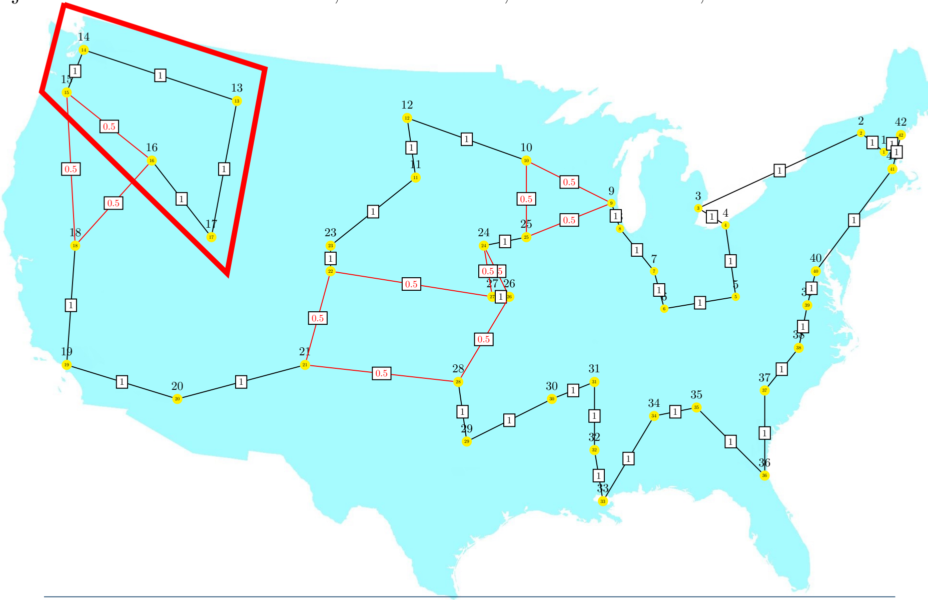
## Iteration 6:

Objective value: $-694.500000$, 861 variables, 950 constraints, 1690 iterations



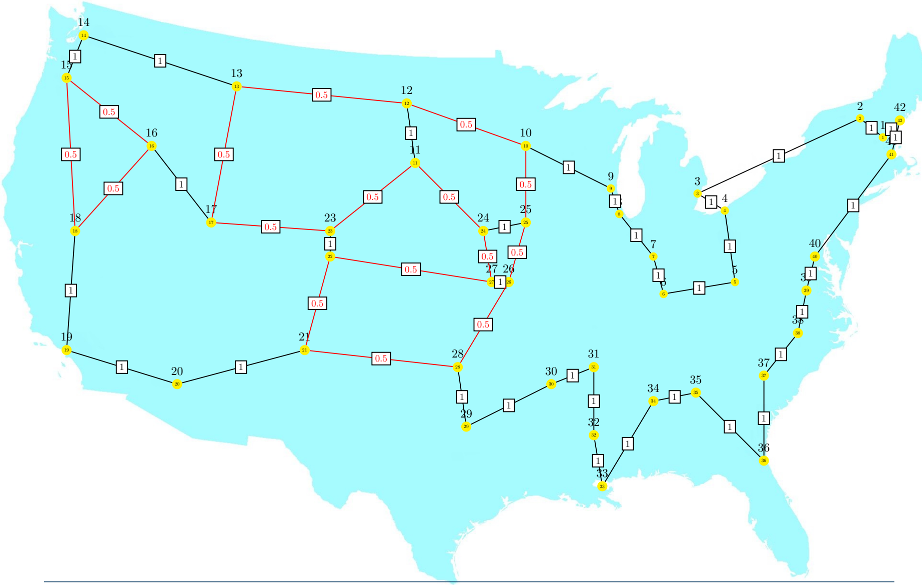
# Iteration 6: Eliminate Cut $13 - 17$

Objective value: $-694.500000$, 861 variables, 950 constraints, 1690 iterations



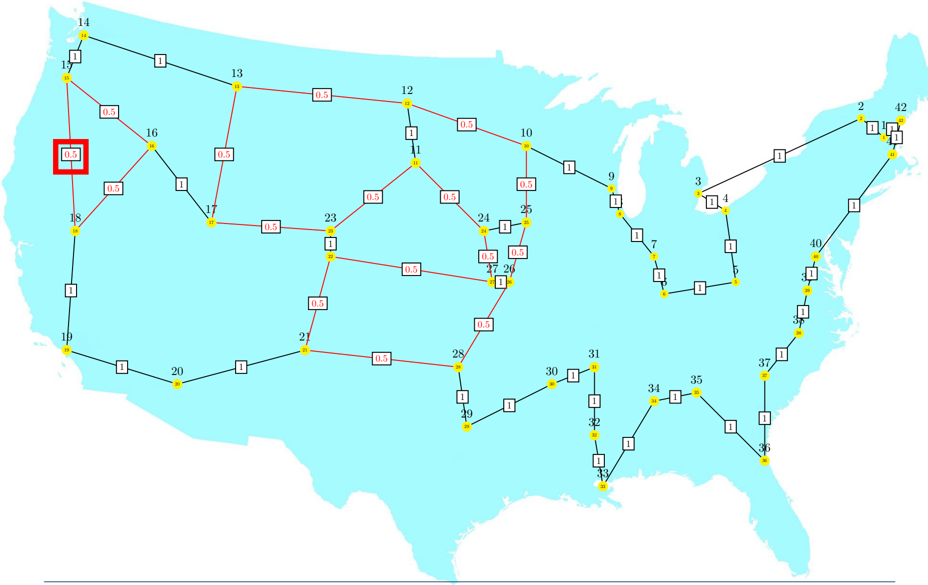
## Iteration 7:

Objective value: $-697.000000$, 861 variables, 951 constraints, 2212 iterations



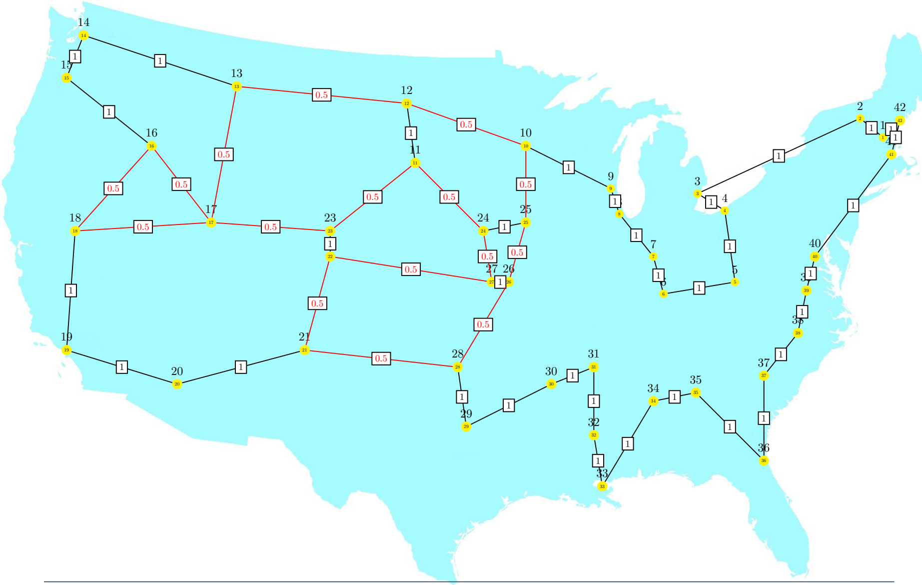
# Iteration 7: Branch 1a $x_{18,15} = 0$

Objective value: $-697.000000$, 861 variables, 951 constraints, 2212 iterations



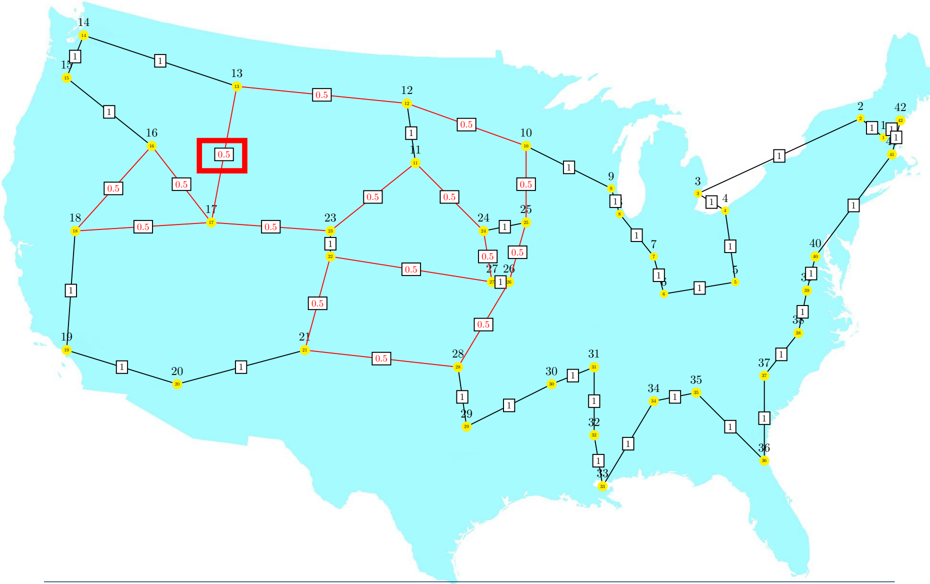
## Iteration 8:

Objective value: $-698.000000$, 861 variables, 952 constraints, 1878 iterations



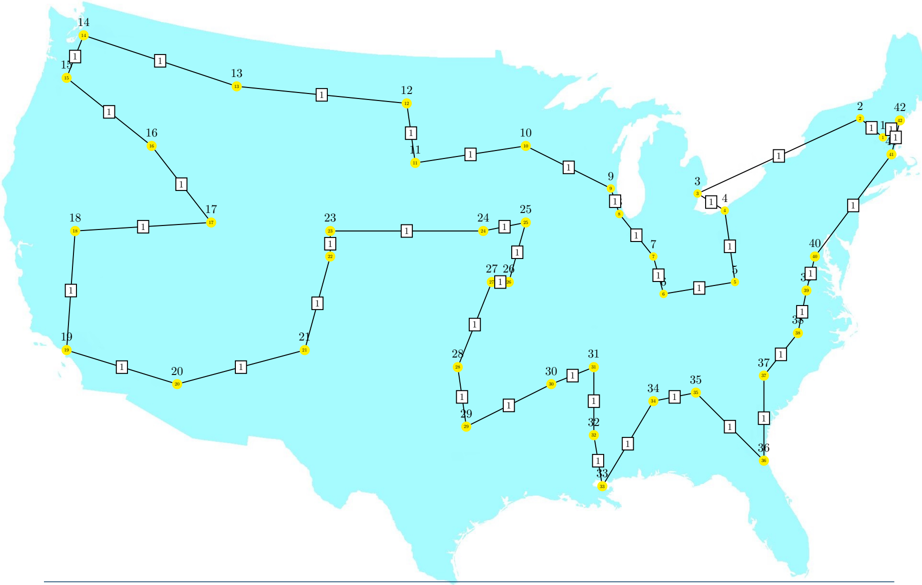
## Iteration 8: Branch 2a $x_{17,13} = 0$

Objective value: $-698.000000$, 861 variables, 952 constraints, 1878 iterations



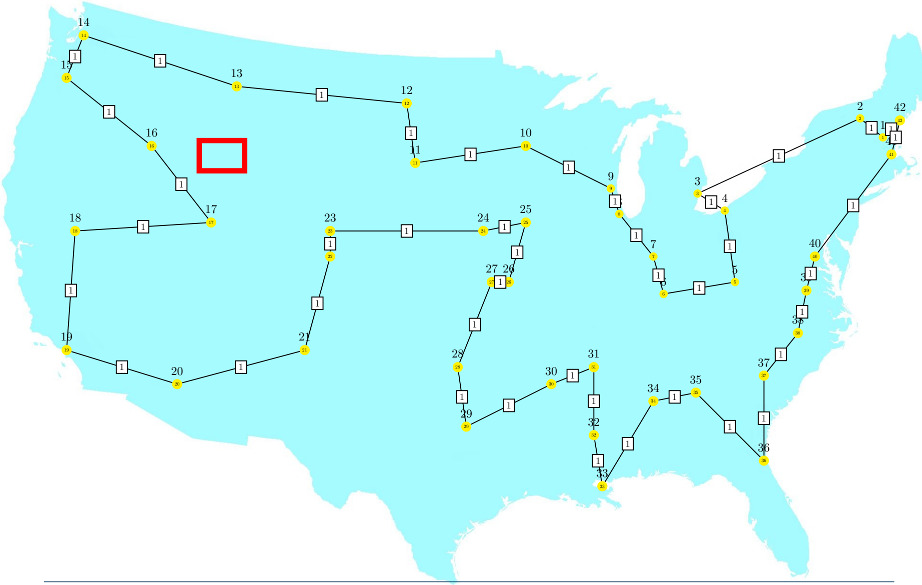
## Iteration 9:

Objective value: $-699.000000$, 861 variables, 953 constraints, 2281 iterations



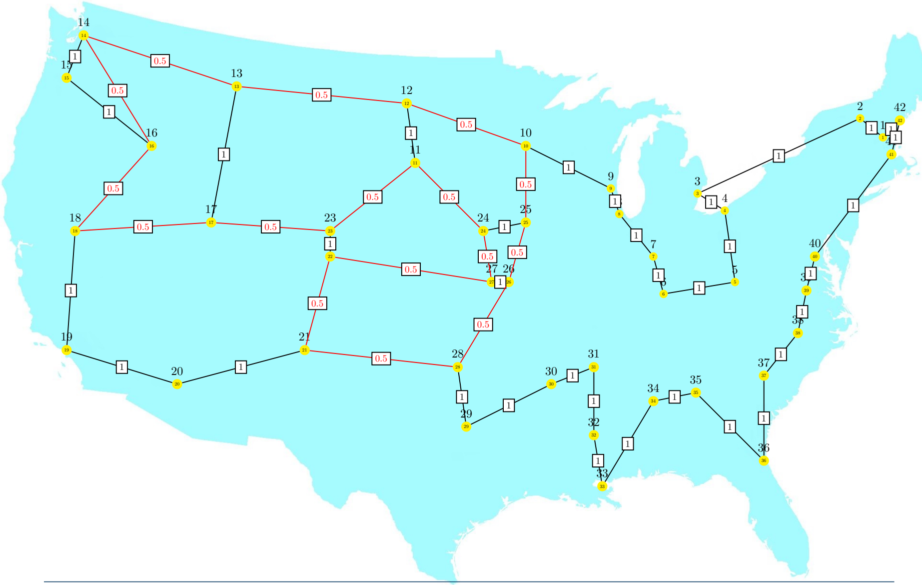
## Iteration 9: Branch 2b $x_{17,13} = 1$

Objective value: $-699.000000$, 861 variables, 953 constraints, 2281 iterations



## Iteration 10:

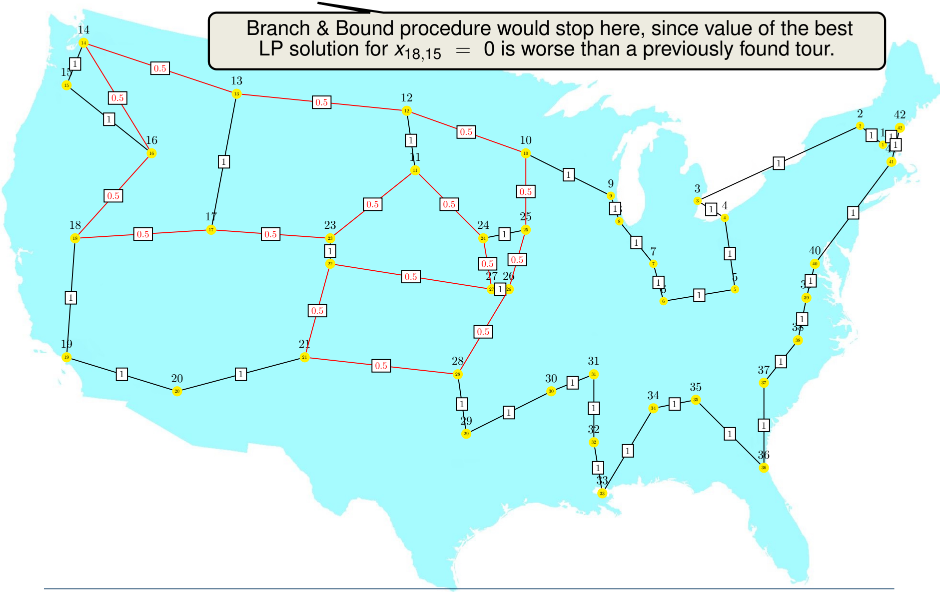
Objective value: $-700.000000$, 861 variables, 954 constraints, 2398 iterations



## Iteration 10:

Objective value: $-700.000000$, 861 variables, 954 constraints, 2398 iterations

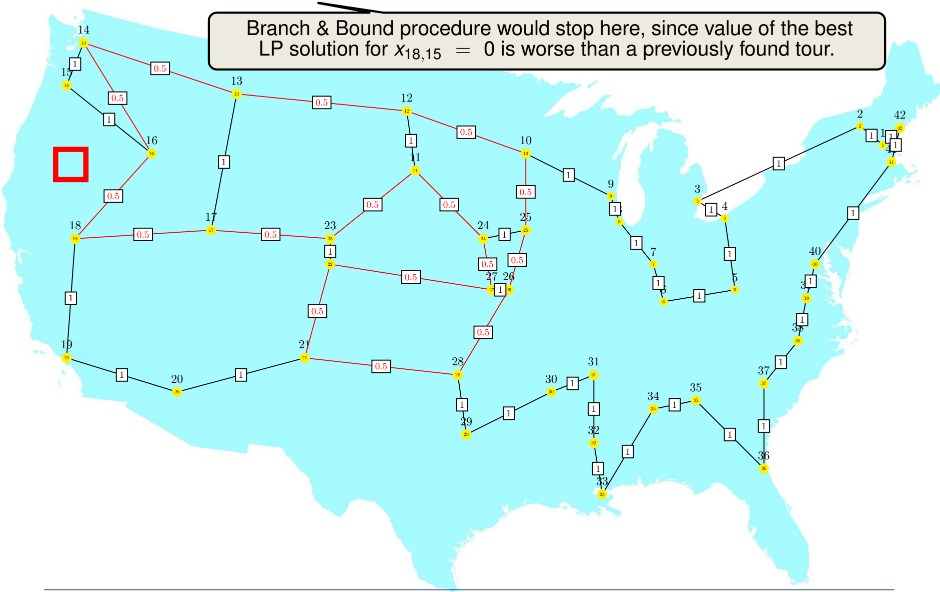
Branch & Bound procedure would stop here, since value of the best LP solution for $x_{18,15} = 0$ is worse than a previously found tour.



## Iteration 10: Branch 1b $x_{18,15} = 1$

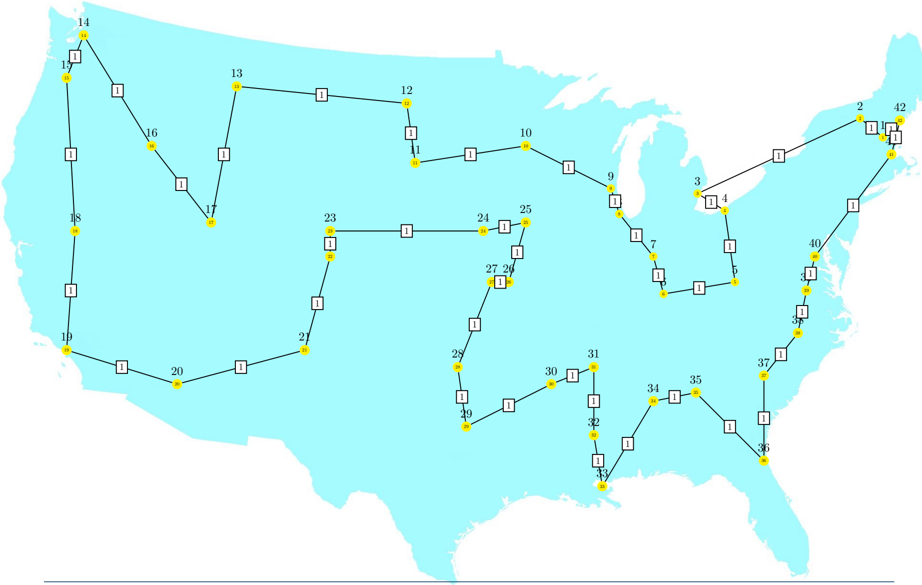
Objective value: $-700.000000$, 861 variables, 954 constraints, 2398 iterations

Branch & Bound procedure would stop here, since value of the best LP solution for $x_{18,15} = 0$ is worse than a previously found tour.



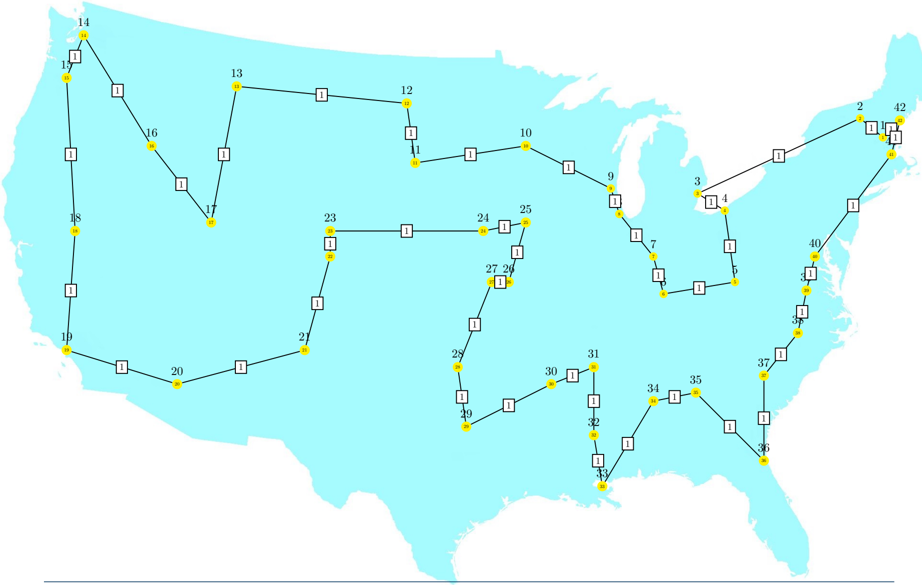
## Iteration 11:

Objective value: $-701.000000$, 861 variables, 953 constraints, 2506 iterations



# Iteration 11: Branch & Bound terminates

Objective value: $-701.000000$, 861 variables, 953 constraints, 2506 iterations



# Branch & Bound Overview

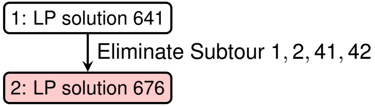
1: LP solution 641

# Branch & Bound Overview

1: LP solution 641

$\downarrow$ Eliminate Subtour $1, 2, 41, 42$

# Branch & Bound Overview



# Branch & Bound Overview

1: LP solution 641

↓ Eliminate Subtour $1, 2, 41, 42$

2: LP solution 676

↓ Eliminate Subtour $3 - 9$

## Branch & Bound Overview

$$
\begin{tikzcd}
\boxed{\text{1: LP solution 641}} \arrow[d, "\text{Eliminate Subtour } 1, 2, 41, 42"] \\
\boxed{\text{2: LP solution 676}} \arrow[d, "\text{Eliminate Subtour } 3 - 9"] \\
\boxed{\text{3: LP solution 681}}
\end{tikzcd}
$$

# Branch & Bound Overview

1: LP solution 641

Eliminate Subtour $1, 2, 41, 42$

2: LP solution 676

Eliminate Subtour $3 - 9$

3: LP solution 681

Eliminate Subtour $24, 25, 26, 27$

# Branch & Bound Overview

1: LP solution 641

↓ Eliminate Subtour $1, 2, 41, 42$

2: LP solution 676

↓ Eliminate Subtour $3 - 9$

3: LP solution 681

↓ Eliminate Subtour $24, 25, 26, 27$

4: LP solution 682.5

# Branch & Bound Overview

1: LP solution 641

↓ Eliminate Subtour $1, 2, 41, 42$

2: LP solution 676

↓ Eliminate Subtour $3 - 9$

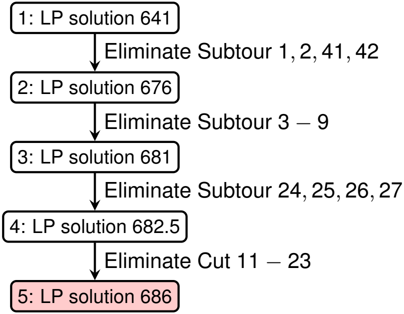
3: LP solution 681

↓ Eliminate Subtour $24, 25, 26, 27$

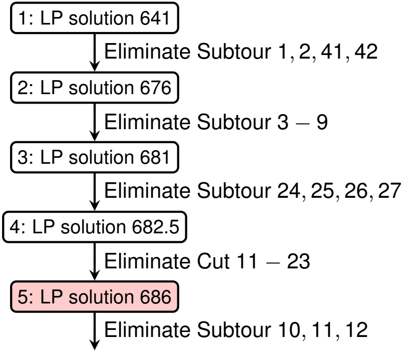
4: LP solution 682.5

↓ Eliminate Cut $11 - 23$

## Branch & Bound Overview



# Branch & Bound Overview



## Branch & Bound Overview

1: LP solution 641

↓ Eliminate Subtour $1, 2, 41, 42$

2: LP solution 676

↓ Eliminate Subtour $3 - 9$

3: LP solution 681

↓ Eliminate Subtour $24, 25, 26, 27$

4: LP solution 682.5

↓ Eliminate Cut $11 - 23$

5: LP solution 686

↓ Eliminate Subtour $10, 11, 12$

6: LP solution 694.5

## Branch & Bound Overview

1: LP solution 641

↓ Eliminate Subtour $1, 2, 41, 42$

2: LP solution 676

↓ Eliminate Subtour $3 - 9$

3: LP solution 681

↓ Eliminate Subtour $24, 25, 26, 27$

4: LP solution 682.5

↓ Eliminate Cut $11 - 23$

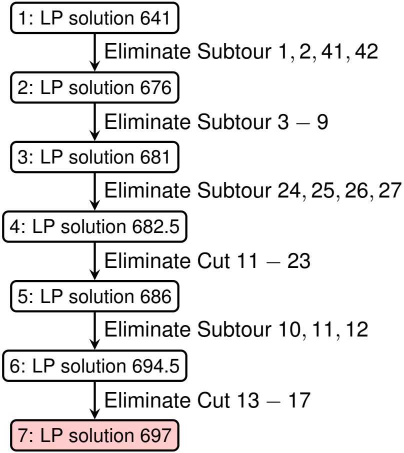
5: LP solution 686

↓ Eliminate Subtour $10, 11, 12$

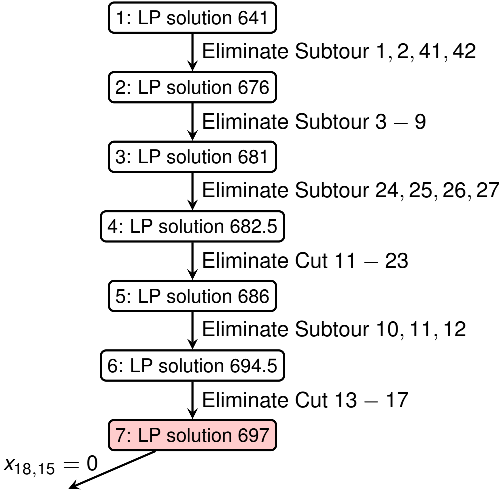
6: LP solution 694.5

↓ Eliminate Cut $13 - 17$

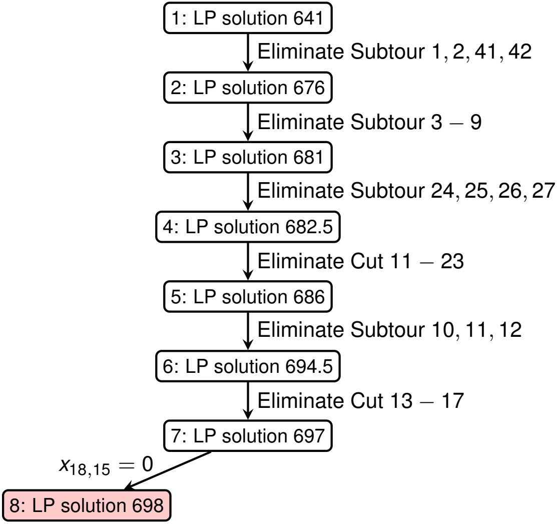
# Branch & Bound Overview



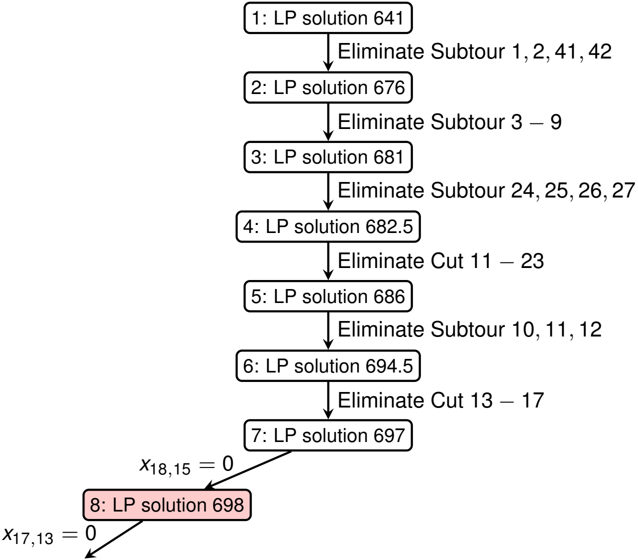
# Branch & Bound Overview



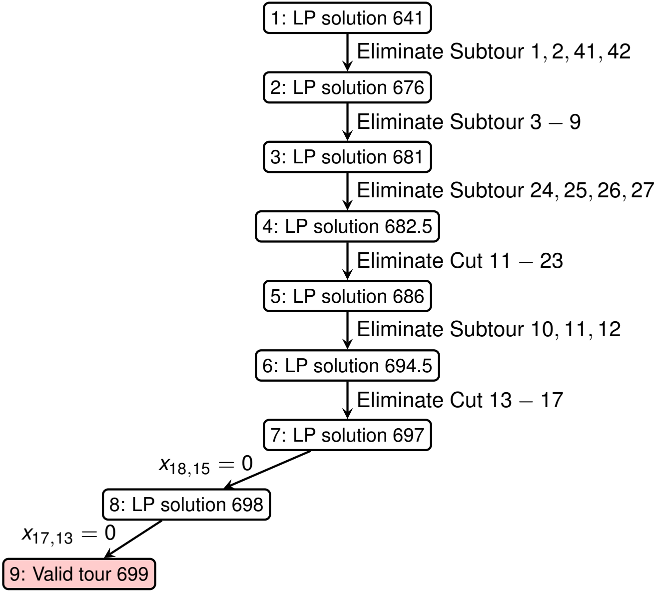
# Branch & Bound Overview



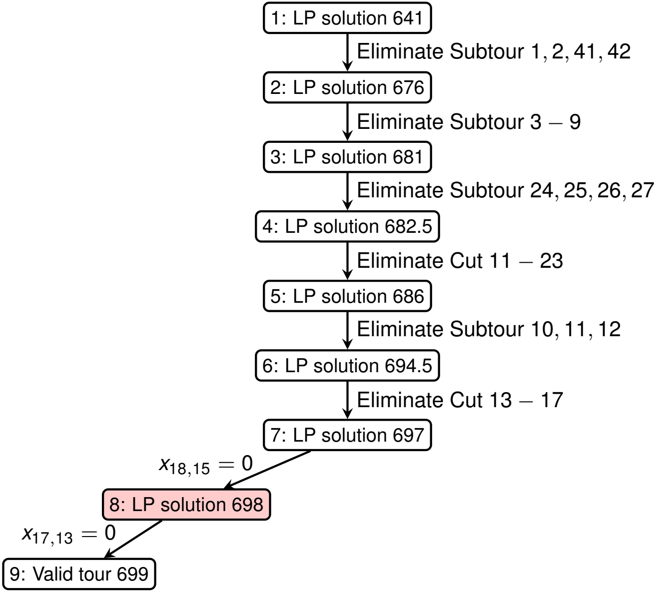
# Branch & Bound Overview



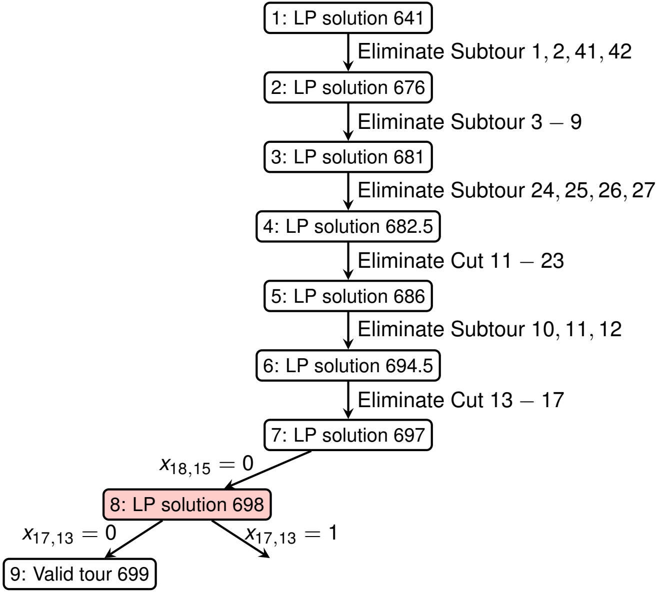
# Branch & Bound Overview



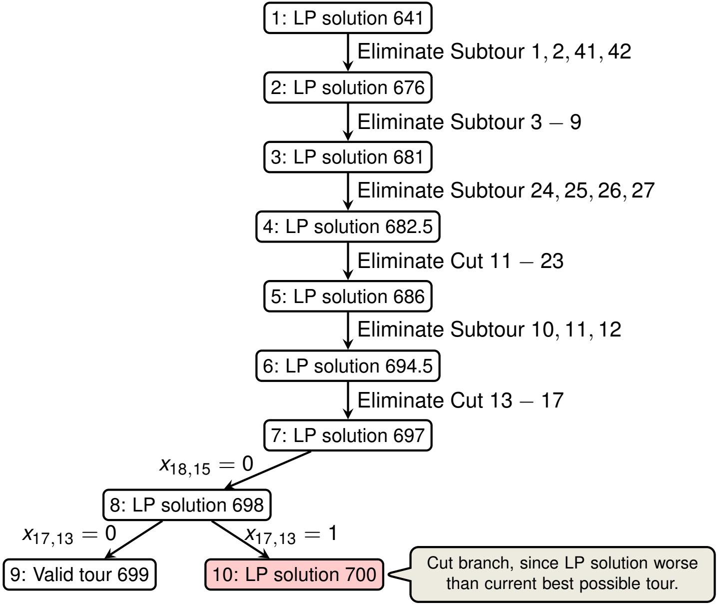
# Branch & Bound Overview



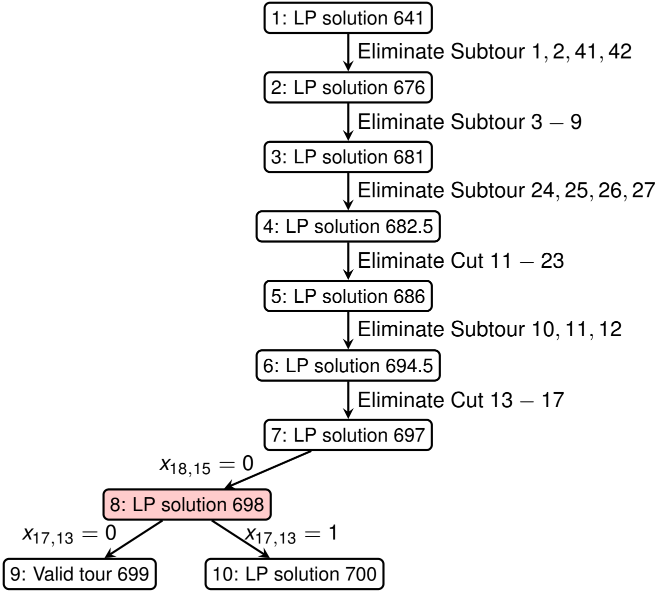
## Branch & Bound Overview



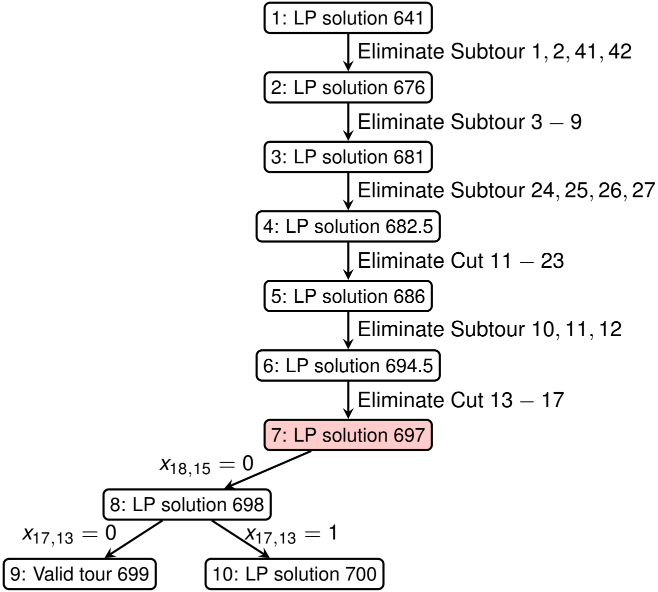
# Branch & Bound Overview



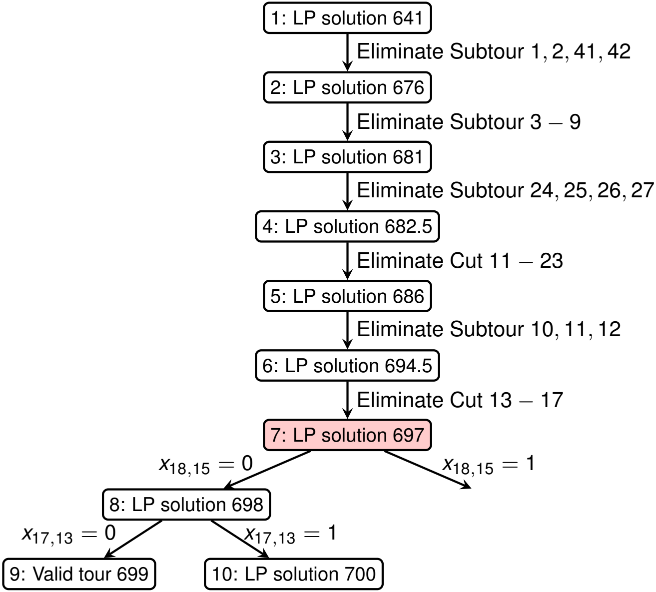
# Branch & Bound Overview



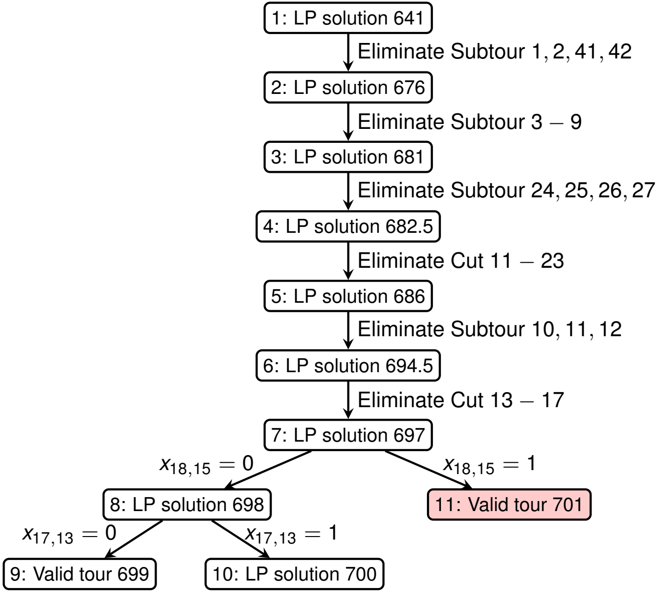
# Branch & Bound Overview



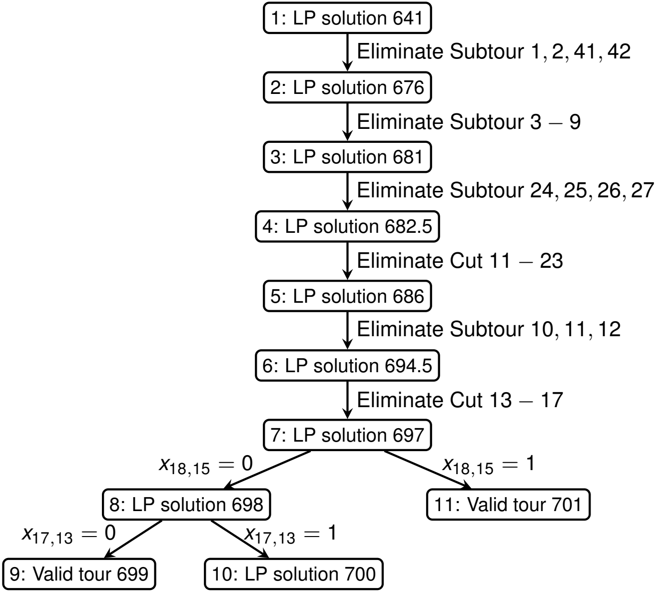
# Branch & Bound Overview



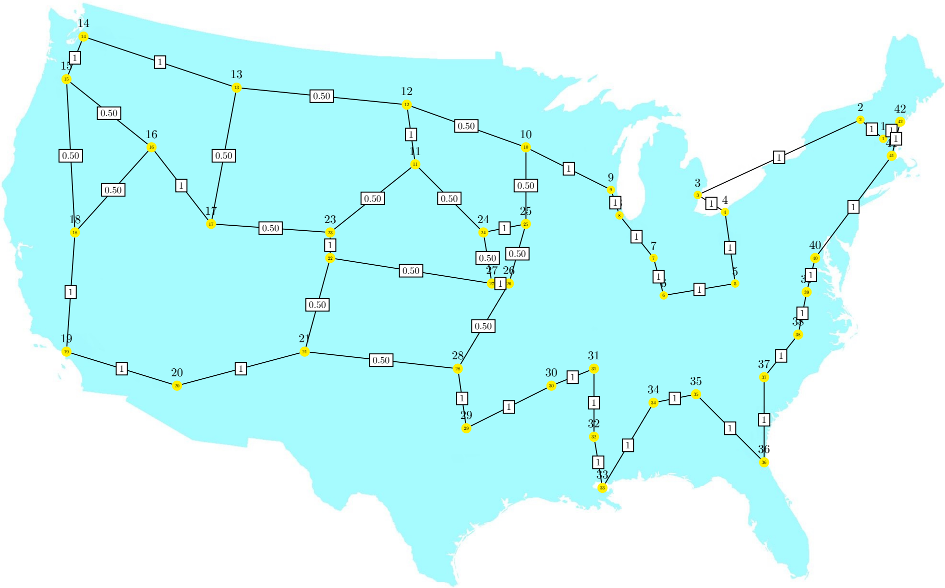
# Branch & Bound Overview



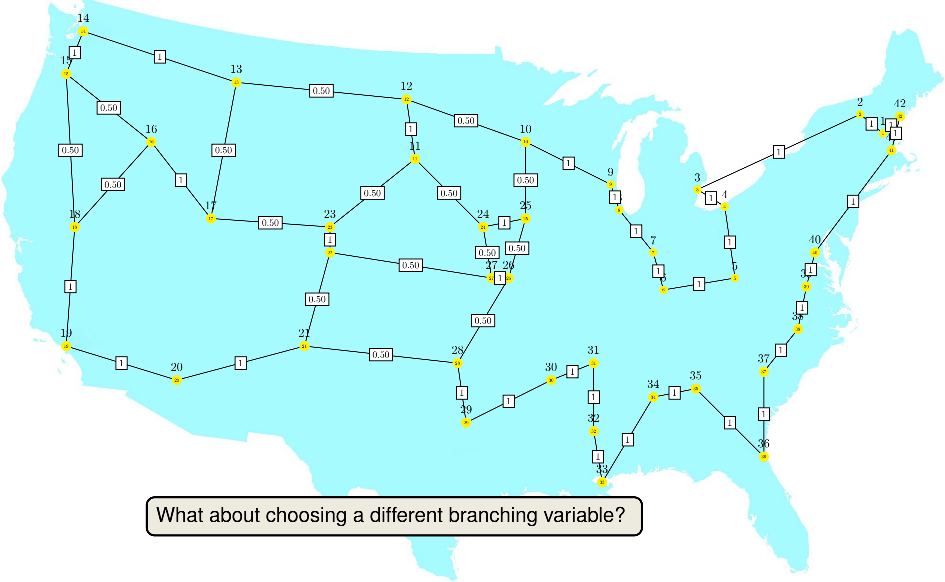
# Branch & Bound Overview



# Iteration 7: Objective 697



# Iteration 7: Objective 697




What about choosing a different branching variable?

# Solving Progress (Alternative Branch 1)

1: LP solution 641

↓ Eliminate Subtour $1, 2, 41, 42$

2: LP solution 676

↓ Eliminate Subtour $3 - 9$

3: LP solution 681

↓ Eliminate Subtour $24, 25, 26, 27$

4: LP solution 682.5

↓ Eliminate Cut $13 - 17$

5: LP solution 686

↓ Eliminate Subtour $10, 11, 12$

6: LP solution 686

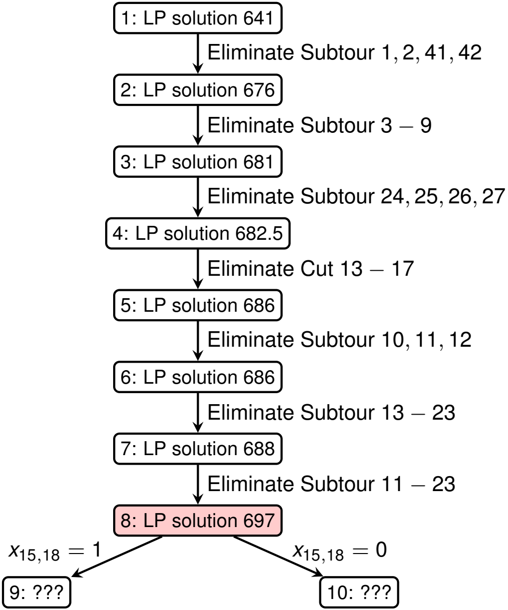
↓ Eliminate Subtour $13 - 23$

7: LP solution 688

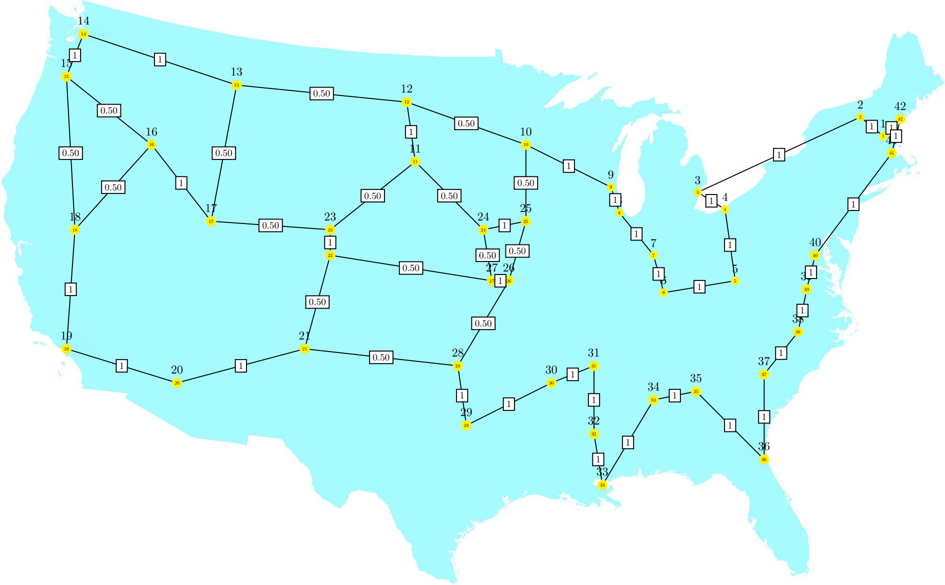
↓ Eliminate Subtour $11 - 23$

8: LP solution 697

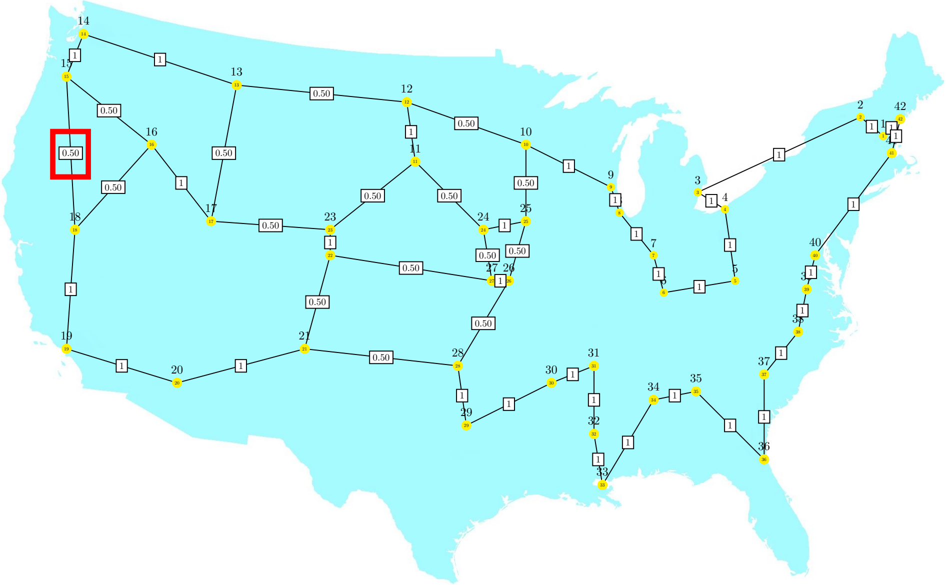
# Solving Progress (Alternative Branch 1)



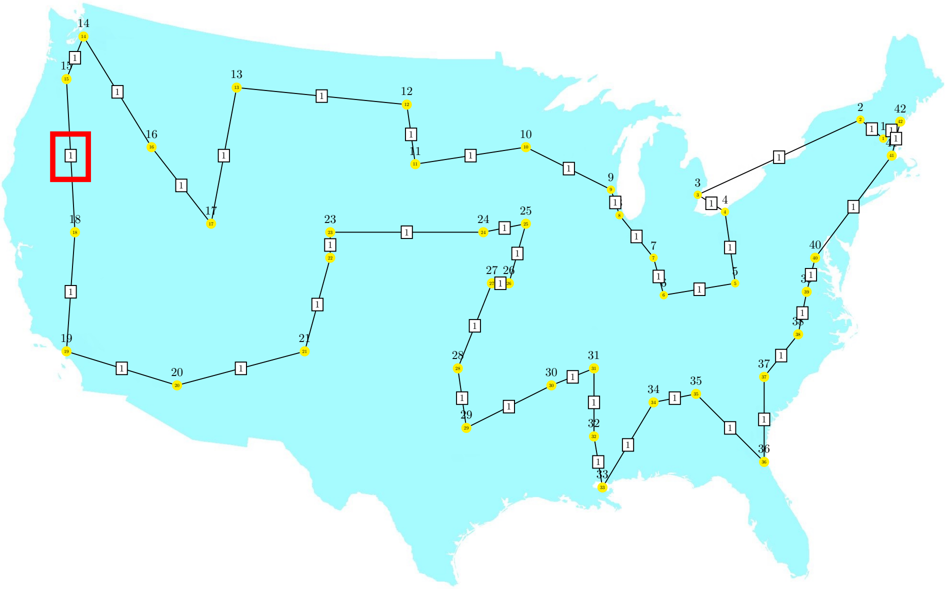
# Alternative Branch 1: $x_{18,15}$, Objective 697



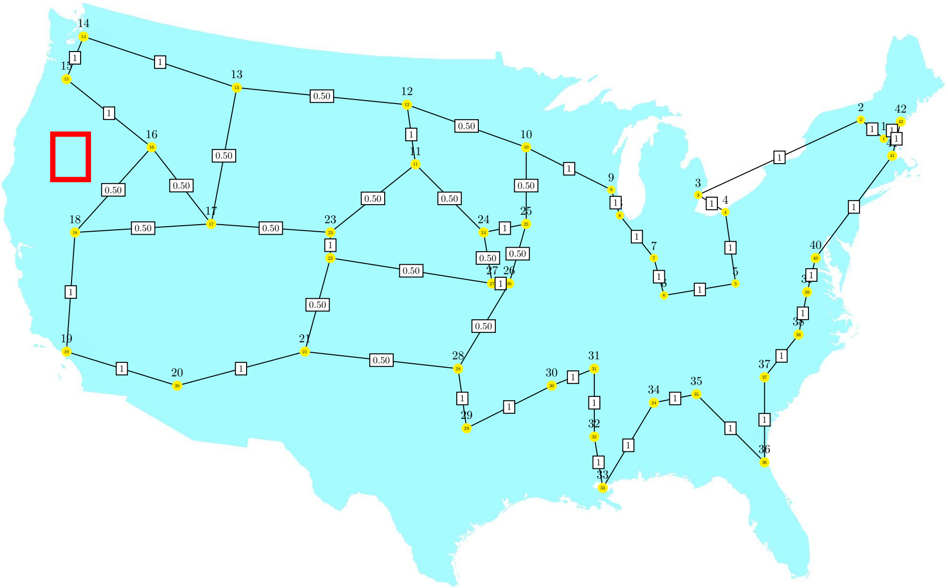
# Alternative Branch 1: $x_{18,15}$, Objective 697



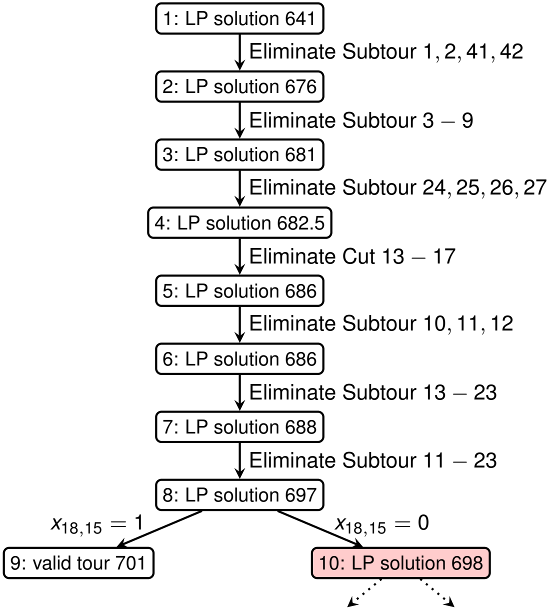
# Alternative Branch 1a: $x_{18,15} = 1$, Objective 701 (Valid Tour)



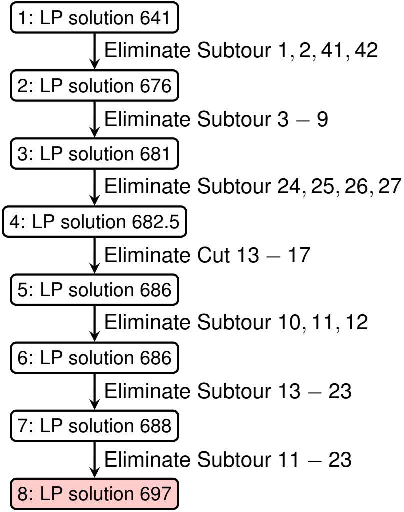
# Alternative Branch 1b: $x_{18,15} = 0$, Objective 698



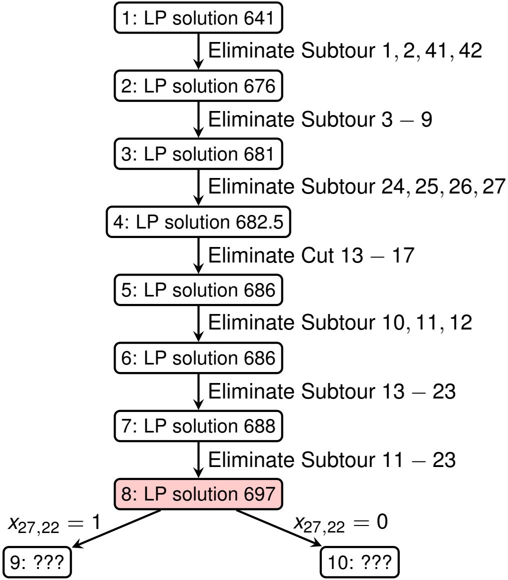
# Solving Progress (Alternative Branch 1)



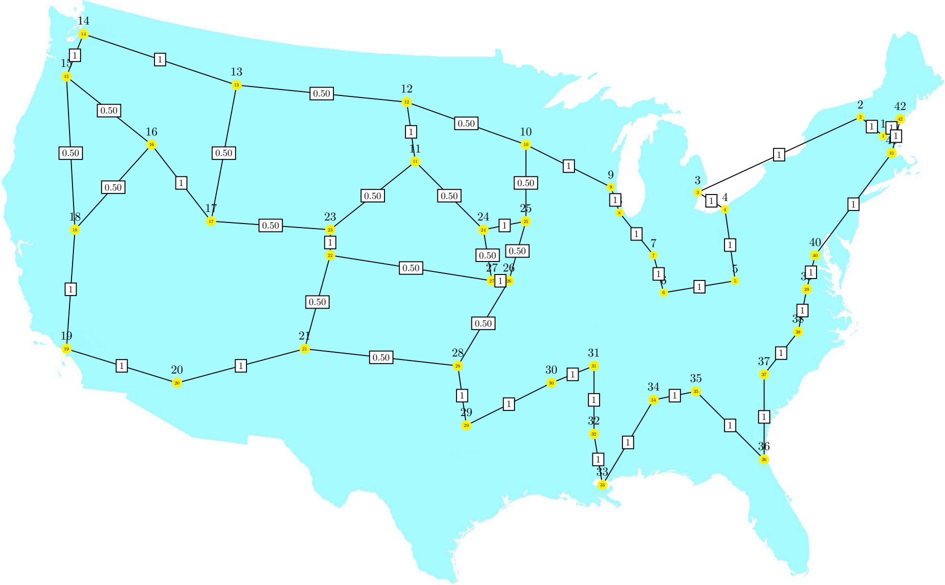
# Solving Progress (Alternative Branch 2)



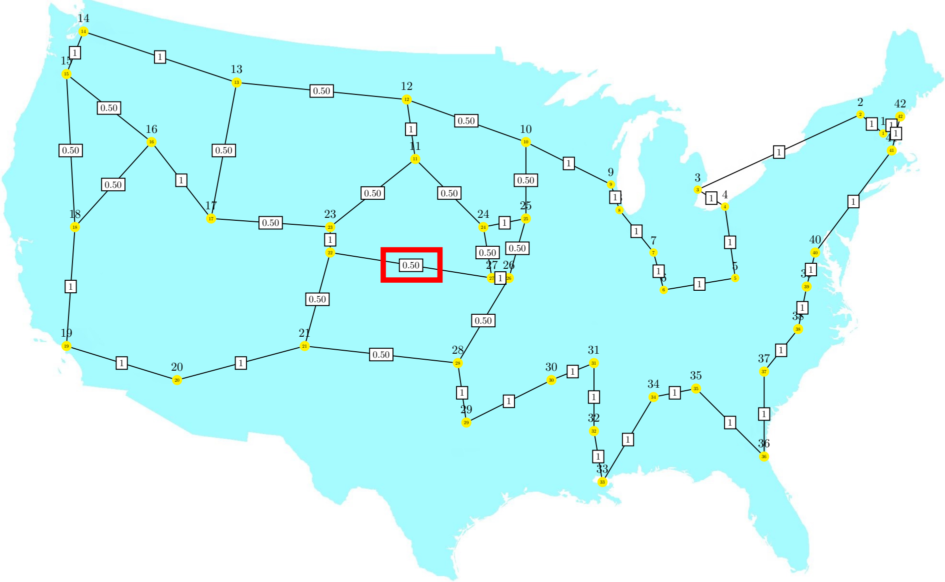
# Solving Progress (Alternative Branch 2)



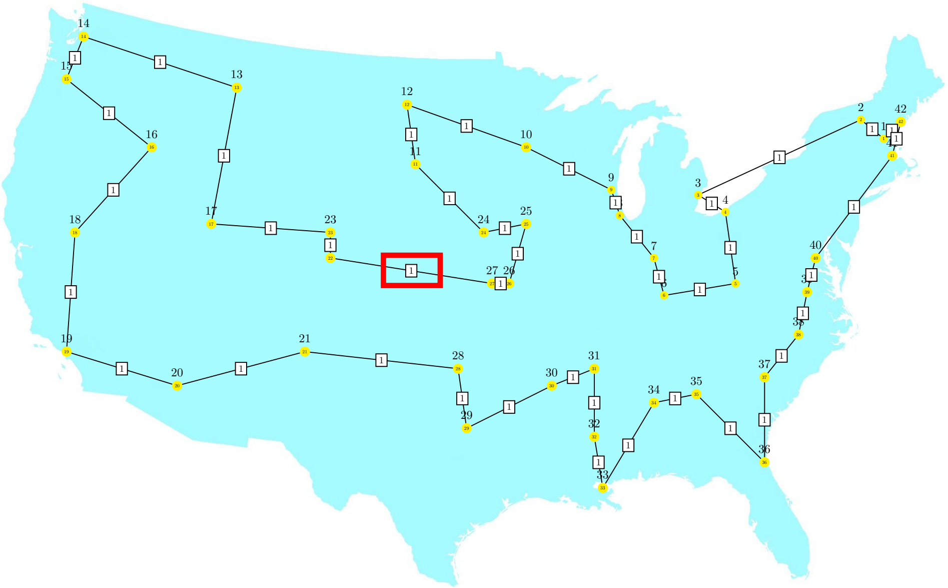
# Alternative Branch 2: $x_{27,22}$, Objective 697



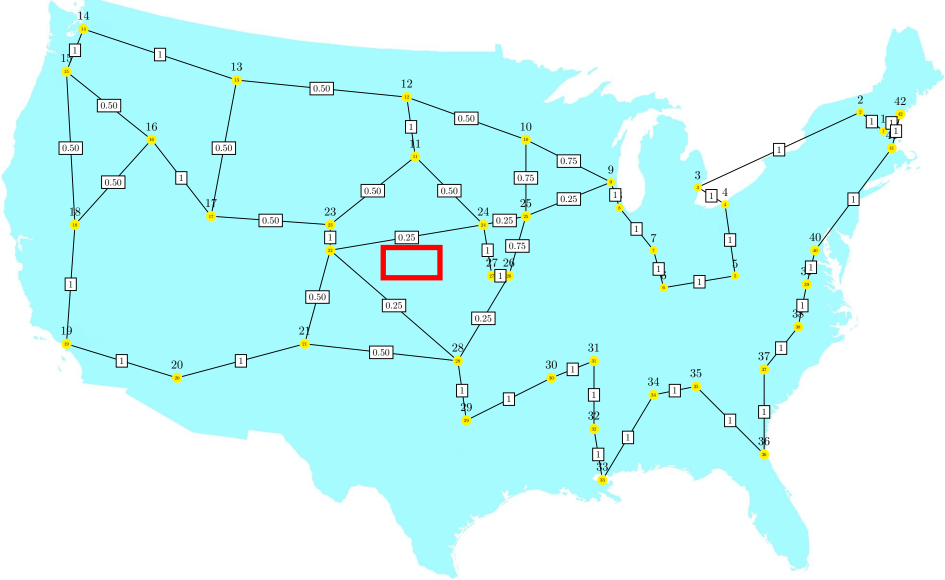
# Alternative Branch 2: $x_{27,22}$, Objective 697



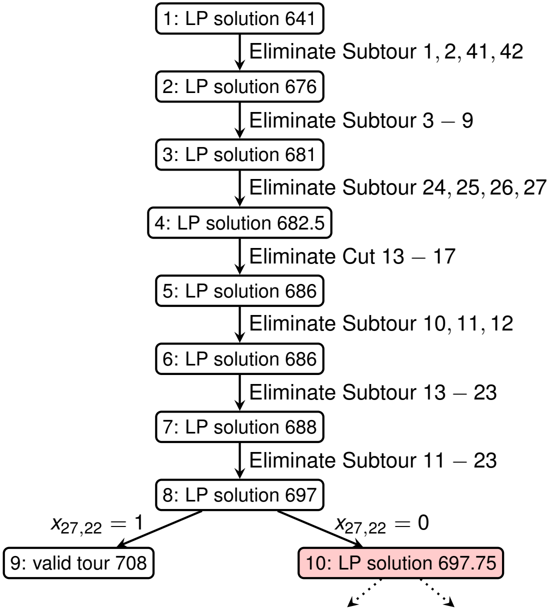
# Alternative Branch 2a: $x_{27,22} = 1$, Objective 708 (Valid tour)



# Alternative Branch 2b: $x_{27,22} = 0$, Objective 697.75



# Solving Progress (Alternative Branch 2)



# Solving Progress (Alternative Branch 3)

1: LP solution 641

↓ Eliminate Subtour $1, 2, 41, 42$

2: LP solution 676

↓ Eliminate Subtour $3 - 9$

3: LP solution 681

↓ Eliminate Subtour $24, 25, 26, 27$

4: LP solution 682.5

↓ Eliminate Cut $13 - 17$

5: LP solution 686

↓ Eliminate Subtour $10, 11, 12$

6: LP solution 686

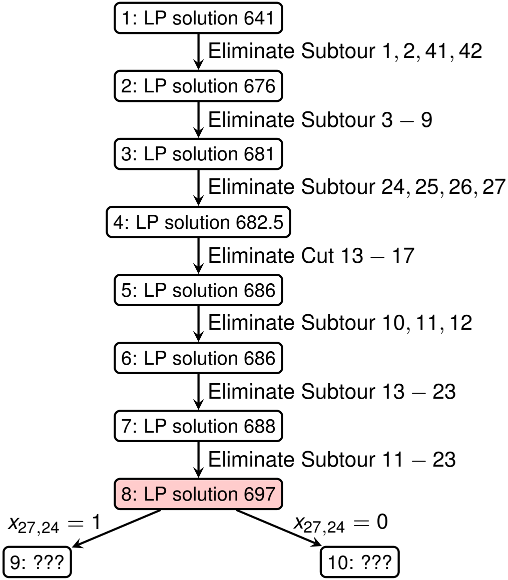
↓ Eliminate Subtour $13 - 23$

7: LP solution 688

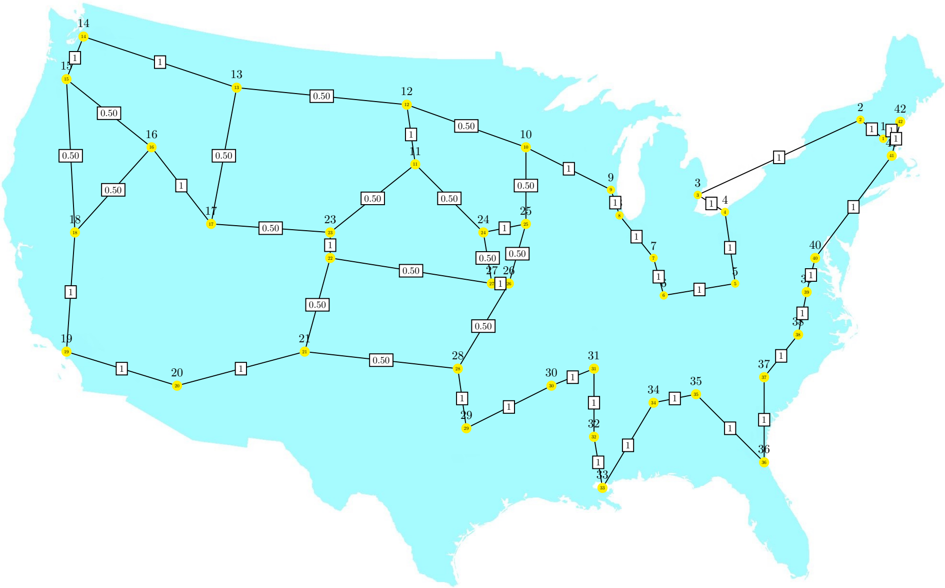
↓ Eliminate Subtour $11 - 23$

8: LP solution 697

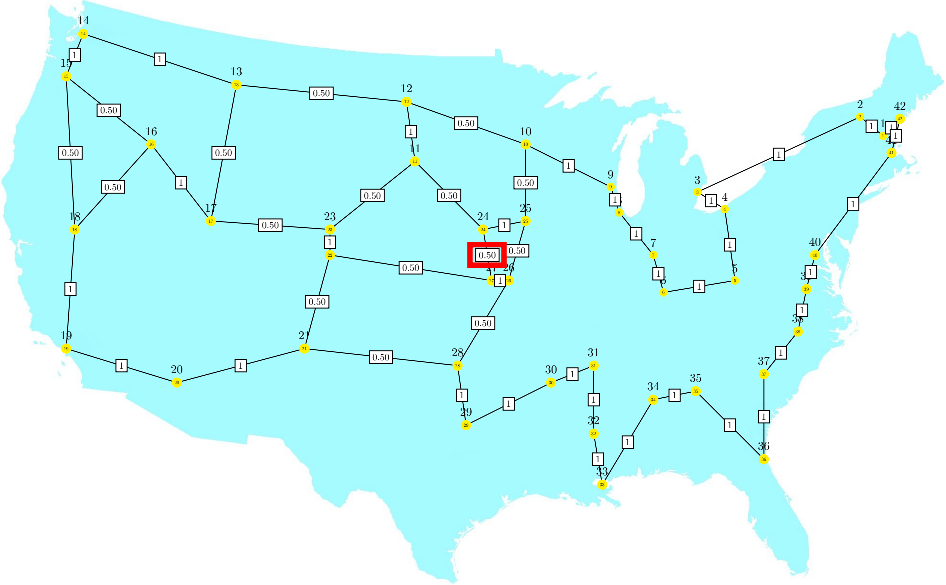
# Solving Progress (Alternative Branch 3)



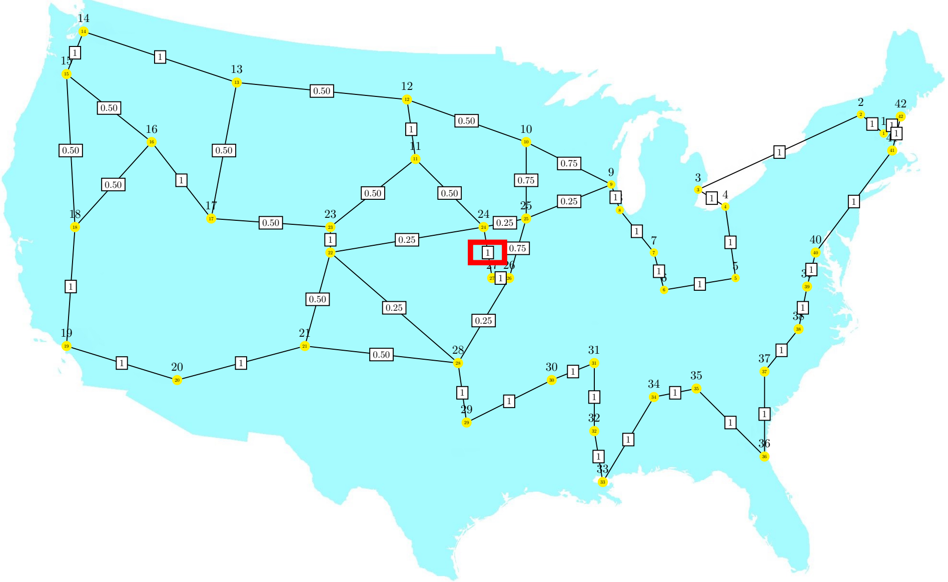
# Alternative Branch 3: $x_{27,24}$, Objective 697



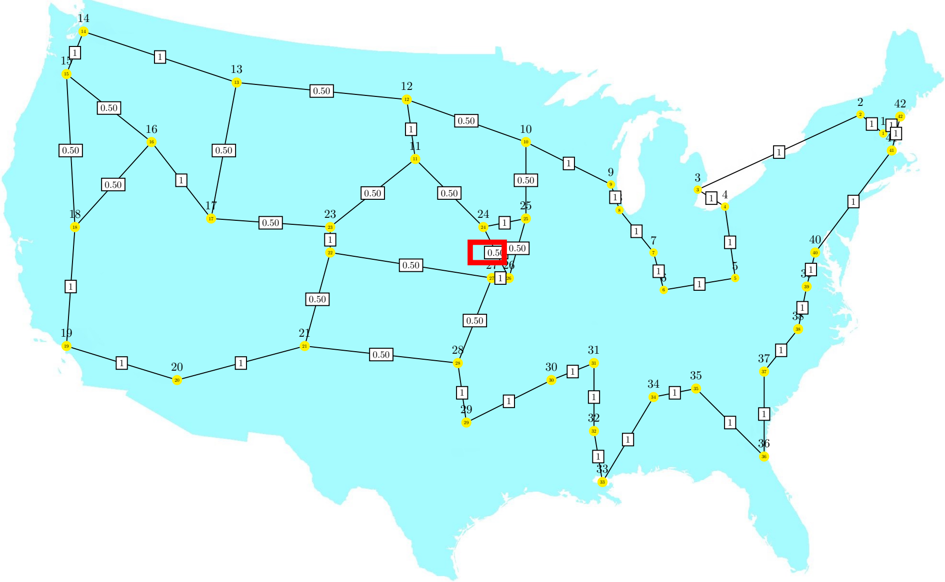
# Alternative Branch 3: $x_{27,24}$, Objective 697



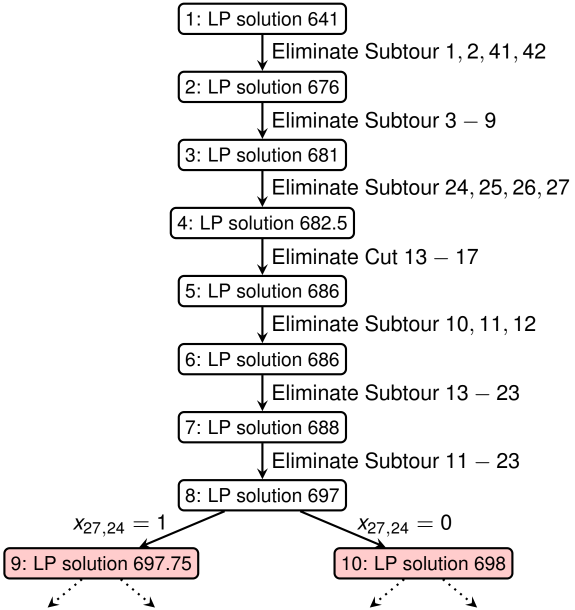
# Alternative Branch 3a: $x_{27,24} = 1$, Objective 697.75



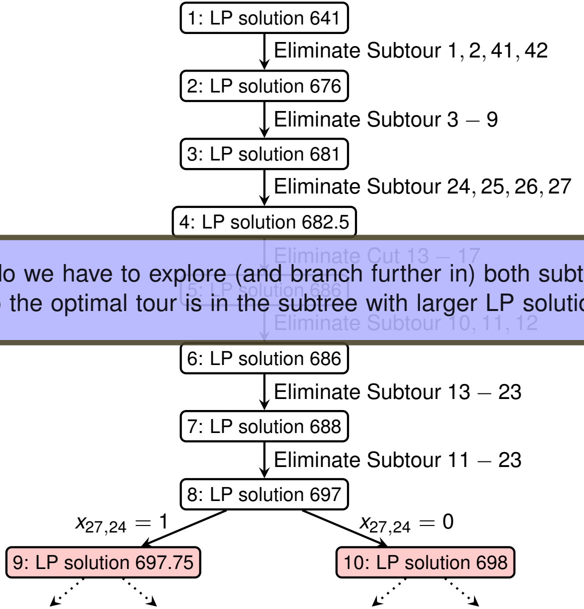
# Alternative Branch 3b: $x_{27,24} = 0$, Objective 698



# Solving Progress (Alternative Branch 3)



# Solving Progress (Alternative Branch 3)




Not only do we have to explore (and branch further in) both subtrees, but also the optimal tour is in the subtree with larger LP solution!

# Conclusion (1/2)

- How can one generate these constraints automatically?

# Conclusion (1/2)

- How can one generate these constraints automatically?
  Subtour Elimination: Finding Connected Components
  Small Cuts: Finding the Minimum Cut in Weighted Graphs

# Conclusion (1/2)

- How can one generate these constraints automatically?
  Subtour Elimination: Finding Connected Components
  Small Cuts: Finding the Minimum Cut in Weighted Graphs

- Why don't we add all possible Subtour Eliminiation constraints to the LP?

## Conclusion (1/2)

- How can one generate these constraints automatically?
  Subtour Elimination: Finding Connected Components
  Small Cuts: Finding the Minimum Cut in Weighted Graphs

- Why don't we add all possible Subtour Eliminiation constraints to the LP?
  There are exponentially many of them!

# Conclusion (1/2)

- How can one generate these constraints automatically?
  Subtour Elimination: Finding Connected Components
  Small Cuts: Finding the Minimum Cut in Weighted Graphs

- Why don't we add all possible Subtour Eliminiation constraints to the LP?
  There are exponentially many of them!

- Should the search tree be explored by BFS or DFS?

## Conclusion (1/2)

- How can one generate these constraints automatically?
  Subtour Elimination: Finding Connected Components
  Small Cuts: Finding the Minimum Cut in Weighted Graphs

- Why don't we add all possible Subtour Eliminiation constraints to the LP?
  There are exponentially many of them!

- Should the search tree be explored by BFS or DFS?
  BFS may be more attractive, even though it might need more memory.

# Conclusion (1/2)

- How can one generate these constraints automatically?
  Subtour Elimination: Finding Connected Components
  Small Cuts: Finding the Minimum Cut in Weighted Graphs

- Why don't we add all possible Subtour Eliminiation constraints to the LP?
  There are exponentially many of them!

- Should the search tree be explored by BFS or DFS?
  BFS may be more attractive, even though it might need more memory.

CONCLUDING REMARK

It is clear that we have left unanswered practically any question one might pose of a theoretical nature concerning the traveling-salesman problem; however, we hope that the feasibility of attacking problems involving a moderate number of points has been successfully demonstrated, and that perhaps some of the ideas can be used in problems of similar nature.

## Conclusion (2/2)

- Eliminate Subtour $1, 2, 41, 42$
- Eliminate Subtour $3 - 9$
- Eliminate Subtour $10, 11, 12$
- Eliminate Subtour $11 - 23$
- Eliminate Subtour $13 - 23$
- Eliminate Cut $13 - 17$
- Eliminate Subtour $24, 25, 26, 27$

# Conclusion (2/2)

- Eliminate Subtour $1, 2, 41, 42$
- Eliminate Subtour $3 - 9$
- Eliminate Subtour $10, 11, 12$
- Eliminate Subtour $11 - 23$
- Eliminate Subtour $13 - 23$
- Eliminate Cut $13 - 17$
- Eliminate Subtour $24, 25, 26, 27$

THE 49-CITY PROBLEM*

The optimal tour $\bar{x}$ is shown in Fig. 16. The proof that it is optimal is given in Fig. 17. To make the correspondence between the latter and its programming problem clear, we will write down in addition to 42 relations in non-negative variables (2), a set of 25 relations which suffice to prove that $D(x)$ is a minimum for $\bar{x}$. We distinguish the following subsets of the 42 cities:

$$S_1 = \{1, 2, 41, 42\} \qquad S_5 = \{13, 14, \cdots, 23\}$$
$$S_2 = \{3, 4, \cdots, 9\} \qquad S_6 = \{13, 14, 15, 16, 17\}$$
$$S_3 = \{1, 2, \cdots, 9, 29, 30, \cdots, 42\} \qquad S_7 = \{24, 25, 26, 27\}.$$
$$S_4 = \{11, 12, \cdots, 23\}$$

# CPLEX

en.wikipedia.org/wiki/CPLEX

WIKIPEDIA
The Free Encyclopedia

- Main page
- Contents
- Featured content
- Current events
- Random article
- Donate to Wikipedia
- Wikipedia store

Interaction

- Help
- About Wikipedia
- Community portal
- Recent changes
- Contact page

Tools

- What links here
- Related changes
- Upload file
- Special pages

## CPLEX

From Wikipedia, the free encyclopedia

**IBM ILOG CPLEX Optimization Studio** (often informally referred to as simply as CPLEX) is an optimization software package. In 2004, the work on CPLEX earned the first INFORMS Impact Prize.

**CPLEX**

| | |
|---|---|
| **Developer(s)** | IBM |
| **Stable release** | 12.6 |
| **Development status** | Active |
| **Type** | Technical computing |
| **License** | Proprietary |
| **Website** | ibm.com/software/products/ibmilogcpleoptistud/ |

The CPLEX Optimizer was named for the simplex method as implemented in the C programming language, although today it also supports other types of mathematical optimization and offers interfaces other than just C. It was originally developed by Robert E. Bixby and was offered commercially starting in 1988 by CPLEX Optimization Inc., which was acquired by ILOG in 1997; ILOG was subsequently acquired by IBM in January 2009.[1] CPLEX continues to be actively developed under IBM.

The IBM ILOG CPLEX Optimizer solves integer programming problems, very large[2] linear programming problems using either primal or dual variants of the simplex method or the barrier interior

```
Welcome to IBM(R) ILOG(R) CPLEX(R) Interactive Optimizer 12.6.1.0
  with Simplex, Mixed Integer & Barrier Optimizers
5725-A06 5725-A29 5724-Y48 5724-Y49 5724-Y54 5724-Y55 5655-Y21
Copyright IBM Corp. 1988, 2014.  All Rights Reserved.

Type 'help' for a list of available commands.
Type 'help' followed by a command name for more
information on commands.

CPLEX> read tsp.lp
Problem 'tsp.lp' read.
Read time = 0.00 sec. (0.06 ticks)
CPLEX> primopt
Tried aggregator 1 time.
LP Presolve eliminated 1 rows and 1 columns.
Reduced LP has 49 rows, 860 columns, and 2483 nonzeros.
Presolve time = 0.00 sec. (0.36 ticks)

Iteration log . . .
Iteration:     1    Infeasibility =            33.999999
Iteration:    26    Objective     =          1510.000000
Iteration:    90    Objective     =           923.000000
Iteration:   155    Objective     =           711.000000

Primal simplex - Optimal:  Objective =  6.9900000000e+02
Solution time =    0.00 sec.  Iterations = 168 (25)
Deterministic time = 1.16 ticks  (288.86 ticks/sec)

CPLEX> 
```

```
CPLEX> display solution variables -
Variable Name           Solution Value
x_2_1                         1.000000
x_42_1                        1.000000
x_3_2                         1.000000
x_4_3                         1.000000
x_5_4                         1.000000
x_6_5                         1.000000
x_7_6                         1.000000
x_8_7                         1.000000
x_9_8                         1.000000
x_10_9                        1.000000
x_11_10                       1.000000
x_12_11                       1.000000
x_13_12                       1.000000
x_14_13                       1.000000
x_15_14                       1.000000
x_16_15                       1.000000
x_17_16                       1.000000
x_18_17                       1.000000
x_19_18                       1.000000
x_20_19                       1.000000
x_21_20                       1.000000
x_22_21                       1.000000
x_23_22                       1.000000
x_24_23                       1.000000
x_25_24                       1.000000
x_26_25                       1.000000
x_27_26                       1.000000
x_28_27                       1.000000
x_29_28                       1.000000
x_30_29                       1.000000
x_31_30                       1.000000
x_32_31                       1.000000
x_33_32                       1.000000
x_34_33                       1.000000
x_35_34                       1.000000
x_36_35                       1.000000
x_37_36                       1.000000
x_38_37                       1.000000
x_39_38                       1.000000
x_40_39                       1.000000
x_41_40                       1.000000
x_42_41                       1.000000
All other variables in the range 1-861 are 0.
```