Randomised Algorithms

Lecture 10: Approximation Algorithms: Set-Cover and MAX-CNF

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UNIVERSITY OF CAMBRIDGE

Outline

Weighted Set Cover

MAX-CNF

The Weighted Set-Cover Problem

Set Cover Problem

- Given: set X and a family of subsets $\mathcal{F}$, and a cost function $c : \mathcal{F} \to \mathbb{R}^+$
- Goal: Find a minimum-cost subset $\mathcal{C} \subseteq \mathcal{F}$

$$\text{s.t.} \qquad X = \bigcup_{S \in \mathcal{C}} S.$$

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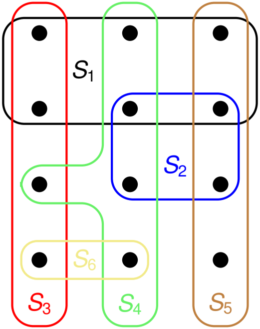
Sum over the costs of all sets in $\mathcal{C}$



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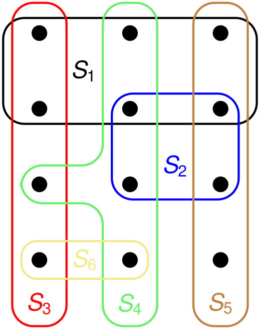

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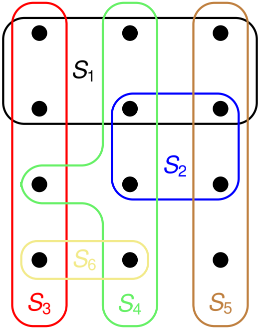

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Remarks:

- generalisation of the weighted Vertex-Cover problem
- models resource allocation problems

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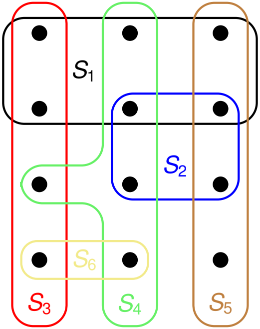

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???

Question: How can we reduce the Vertex-Cover problem to the Set-Cover problem?

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- generalisation of the weighted Vertex-Cover problem
- models resource allocation problems

Setting up an Integer Program

Question: Try to formulate the integer program and linear program of the weighted SET-COVER problem
(solution on next slide!)

Setting up an Integer Program

0-1 Integer Program

$$
\begin{array}{llcll}
\text{minimize} & \displaystyle\sum_{S \in \mathcal{F}} c(S) y(S) & & & \\
\text{subject to} & \displaystyle\sum_{S \in \mathcal{F}\colon x \in S} y(S) & \geq & 1 & \text{for each } x \in X \\
& y(S) & \in & \{0, 1\} & \text{for each } S \in \mathcal{F}
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Linear Program

$$\begin{array}{llcll}
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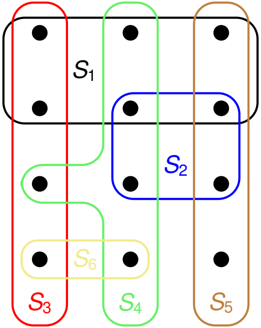
Back to the Example



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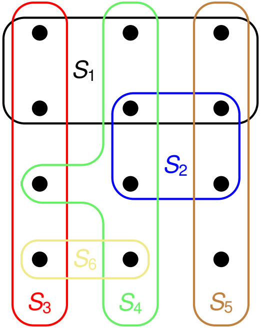

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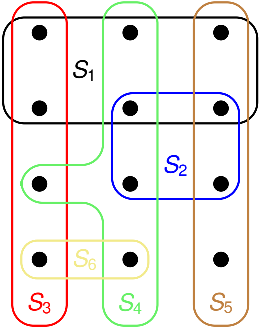

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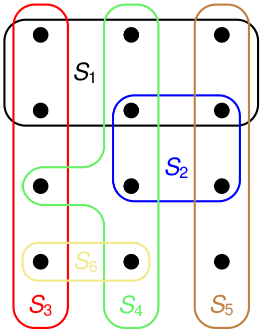

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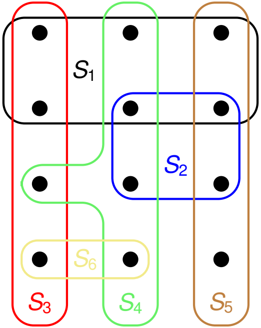

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The strategy employed for Vertex-Cover would take all 6 sets!

Even worse: If all $\overline{y}$'s were below $1/2$, we would not even return a valid cover!

Randomised Rounding

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- Let $\mathcal{C} \subseteq \mathcal{F}$ be a random set with each set S being included independently with probability $\overline{y}(S)$.
- More precisely, if $\overline{y}$ denotes the optimal solution of the LP, then we compute an integral solution y by:

$$y(S) = \begin{cases} 1 & \text{with probability } \overline{y}(S) \\ 0 & \text{otherwise.} \end{cases} \qquad \text{for all } S \in \mathcal{F}.$$

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- Therefore, $\mathbf{E}\left[\, y(S) \,\right] = \overline{y}(S)$.

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- The probability that an element $x \in X$ is covered satisfies

$$\mathbf{P}\left[x \in \bigcup_{S \in \mathcal{C}} S \right] \geq 1 - \frac{1}{e}.$$

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Lemma

Let $\mathcal{C} \subseteq \mathcal{F}$ be a random subset with each set S being included independently with probability $\overline{y}(S)$.

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$$\mathbf{E}[c(\mathcal{C})] = \mathbf{E}\left[\sum_{S \in \mathcal{C}} c(S)\right] = \mathbf{E}\left[\sum_{S \in \mathcal{F}} \mathbf{1}_{S \in \mathcal{C}} \cdot c(S)\right]$$
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- **Step 2**: The probability for an element to be (not) covered

$$\mathbf{P}[x \notin \cup_{S \in \mathcal{C}} S] = \prod_{S \in \mathcal{F}:\, x \in S} \mathbf{P}[S \notin \mathcal{C}] = \prod_{S \in \mathcal{F}:\, x \in S} (1 - \overline{y}(S))$$
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$1 + x \leq e^x$ for any $x \in \mathbb{R}$

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WEIGHTED SET COVER-LP$(X, \mathcal{F}, c)$

1: compute $\overline{y}$, an optimal solution to the linear program
2: $\mathcal{C} = \emptyset$
3: **repeat** $2 \ln n$ times
4: **for** each $S \in \mathcal{F}$
5: let $\mathcal{C} = \mathcal{C} \cup \{S\}$ with probability $\overline{y}(S)$
6: **return** $\mathcal{C}$

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```
compute ȳ, an optimal solution to the linear program
C = ∅
repeat 2 ln n times
    for each S ∈ F
        let C = C ∪ {S} with probability ȳ(S)
return C
```

clearly runs in polynomial-time!

Analysis of WEIGHTED SET COVER-LP

Theorem

- With probability at least $1 - \frac{1}{n}$, the returned set $\mathcal{C}$ is a valid cover of X.
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Analysis of WEIGHTED SET COVER-LP

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- With probability at least $1 - \frac{1}{n}$, the returned set $\mathcal{C}$ is a valid cover of X.
- The expected approximation ratio is $2\ln(n)$.

Proof:

- **Step 1**: The probability that $\mathcal{C}$ is a cover ✓
  - By previous Lemma, an element $x \in X$ is covered in one of the $2\ln n$ iterations with probability at least $1 - \frac{1}{e}$, so that

$$\mathbf{P}\left[\, x \notin \cup_{S \in \mathcal{C}} S \,\right] \leq \left(\frac{1}{e}\right)^{2\ln n} = \frac{1}{n^2}.$$

  - This implies for the event that all elements are covered:

$$\mathbf{P}\left[\, X = \cup_{S \in \mathcal{C}} S \,\right] = 1 - \mathbf{P}\left[\bigcup_{x \in X} \{x \notin \cup_{S \in \mathcal{C}} S\}\right]$$

$\mathbf{P}[\, A \cup B \,] \leq \mathbf{P}[\, A \,] + \mathbf{P}[\, B \,]$

$$\geq 1 - \sum_{x \in X} \mathbf{P}\left[\, x \notin \cup_{S \in \mathcal{C}} S \,\right] \geq 1 - n \cdot \frac{1}{n^2} = 1 - \frac{1}{n}.$$

- **Step 2**: The expected approximation ratio ✓
  - By previous lemma, the expected cost of one iteration is $\sum_{S \in \mathcal{F}} c(S) \cdot \overline{y}(S)$.
  - Linearity $\Rightarrow \mathbf{E}\left[\, c(\mathcal{C}) \,\right] \leq 2\ln(n) \cdot \sum_{S \in \mathcal{F}} c(S) \cdot \overline{y}(S) \leq 2\ln(n) \cdot c(\mathcal{C}^*)$

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[Exercise Question (9/10).10] gives a different perspective on the amplification procedure through non-linear randomised rounding.

Outline

Weighted Set Cover

MAX-CNF

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Recall:

MAX-3-CNF Satisfiability

- Given: 3-CNF formula, e.g.: $(x_1 \lor x_3 \lor \overline{x_4}) \land (x_2 \lor \overline{x_3} \lor \overline{x_5}) \land \cdots$
- Goal: Find an assignment of the variables that satisfies as many clauses as possible.

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Why study this generalised problem?

- Allowing arbitrary clause lengths makes the problem more interesting (we will see that simply guessing is not the best!)
- a nice concluding example where we can practice previously learned approaches

Approach 1: Guessing the Assignment

Assign each variable true or false uniformly and independently at random.

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Proof:

- First statement as in the proof of Theorem 35.6. For clause i not to be satisfied, all ℓ occurring variables must be set to a specific value.
- As before, let $Y := \sum_{i=1}^{m} Y_i$ be the number of satisfied clauses. Then,

$$\mathbf{E}[\,Y\,] = \mathbf{E}\left[\sum_{i=1}^{m} Y_i\right] = \sum_{i=1}^{m} \mathbf{E}[\,Y_i\,] \geq \sum_{i=1}^{m} \frac{1}{2} = \frac{1}{2} \cdot m. \qquad \square$$

Approach 2: Guessing with a "Hunch" (Randomised Rounding)

First solve a linear program and use fractional values for a **biased** coin flip.

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0-1 Integer Program

$$\begin{array}{llcll} \text{maximize} & \displaystyle\sum_{i=1}^{m} z_i & & & \\ \text{subject to} & \displaystyle\sum_{j \in C_i^+} y_j + \sum_{j \in C_i^-} (1 - y_j) & \geq & z_i & \text{for each } i = 1, 2, \ldots, m \\ & z_i & \in & \{0, 1\} & \text{for each } i = 1, 2, \ldots, m \\ & y_j & \in & \{0, 1\} & \text{for each } j = 1, 2, \ldots, n \end{array}$$

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C_i^+ is the index set of the un-negated variables of clause i.

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- In the corresponding LP each $\in \{0, 1\}$ is replaced by $\in [0, 1]$
- Let $(\overline{y}, \overline{z})$ be the optimal solution of the LP
- Obtain an integer solution y through randomised rounding of $\overline{y}$

Analysis of Randomised Rounding

Lemma

For any clause i of length ℓ,

$$\mathbf{P}\left[\,\text{clause } i \text{ is satisfied}\,\right] \geq \left(1 - \left(1 - \frac{1}{\ell}\right)^{\ell}\right) \cdot \overline{z}_i.$$

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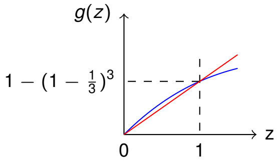

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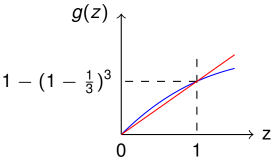

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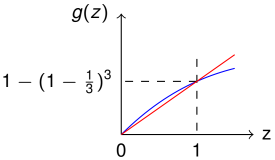

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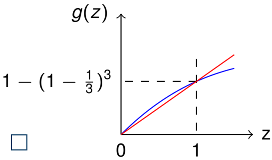

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LP solution at least as good as optimum

Approach 3: Hybrid Algorithm

Summary

- Approach 1 (Guessing) achieves better guarantee on longer clauses
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HYBRID-MAX-CNF(φ, n, m)

Let $b \in \{0, 1\}$ be the flip of a fair coin
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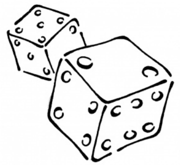

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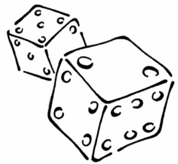

Algorithm sets each variable x_i to TRUE with prob. $\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \overline{y}_i$.
Note, however, that variables are **not** independently assigned!

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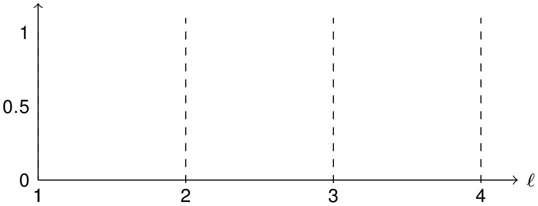

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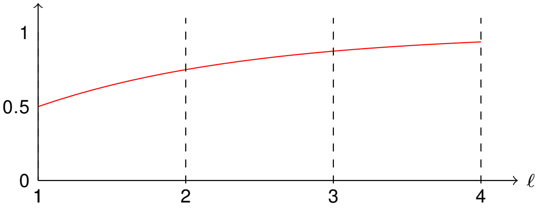

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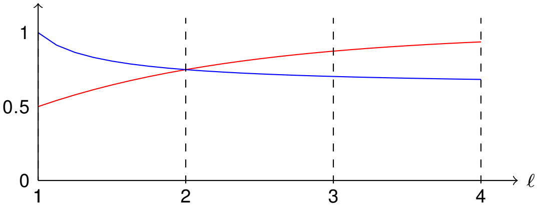

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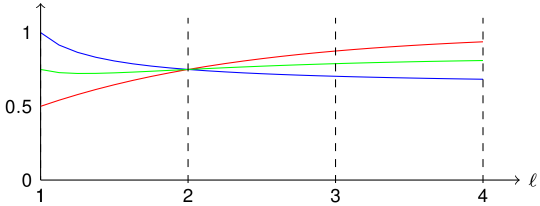

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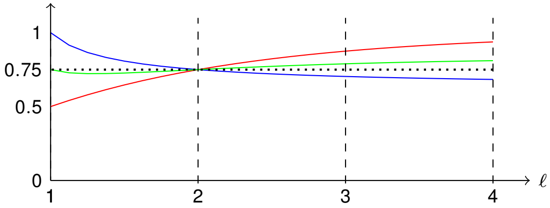

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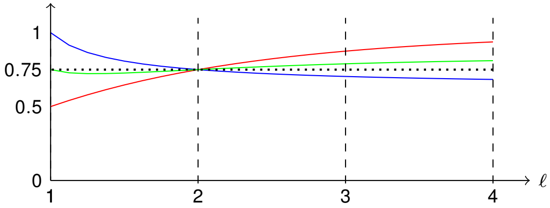
- $\Rightarrow$ HYBRID-MAX-CNF(φ, n, m) satisfies it with prob. at least $3/4 \cdot \overline{z}_i$ □



MAX-CNF Conclusion

Summary

- Since $\alpha_2 = \beta_2 = 3/4$, we cannot achieve a better approximation ratio than $4/3$ by combining Algorithm 1 & 2 in a different way
- The $4/3$-approximation algorithm can be easily derandomised
  - Idea: use the conditional expectation trick for both Algorithm 1 & 2 and output the better solution
- The $4/3$-approximation algorithm applies unchanged to a weighted version of MAX-CNF, where each clause has a non-negative weight
- Even MAX-2-CNF (every clause has length 2) is NP-hard!