Randomised Algorithms

Lecture 6: Linear Programming: Introduction

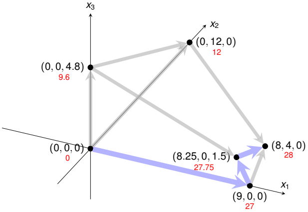
Thomas Sauerwald (tms41@cam.ac.uk)

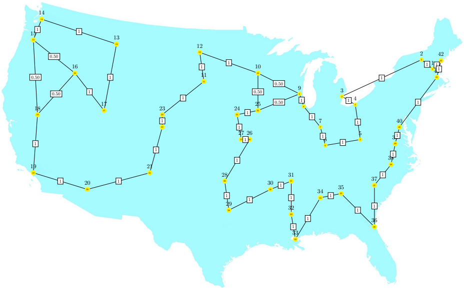
Lent 2024

UNIVERSITY OF CAMBRIDGE

Outline

Introduction





- linear programming is a powerful tool in optimisation
- inspired more sophisticated techniques such as quadratic optimisation, convex optimisation, integer programming and semi-definite programming
- we will later use the connection between linear and integer programming to tackle several problems (Vertex-Cover, Set-Cover, TSP, satisfiability)

Outline

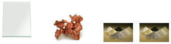
What are Linear Programs?

Linear Programming (informal definition)

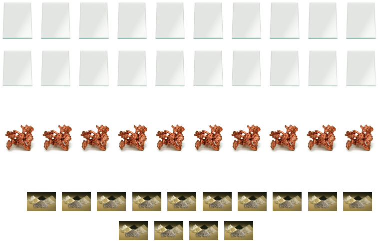
- maximise or minimise an objective, given limited resources (competing constraint)
- constraints are specified as (in)equalities
- objective function and constraints are linear

A Simple Example of a Linear Optimisation Problem

- Laptop
  - selling price to retailer: 1,000 GBP
  - glass: 4 units
  - copper: 2 units
  - rare-earth elements: 1 unit



- Smartphone
  - selling price to retailer: 1,000 GBP
  - glass: 1 unit
  - copper: 1 unit
  - rare-earth elements: 2 units



- You have a daily supply of:
  - glass: 20 units
  - copper: 10 units
  - rare-earth elements: 14 units
  - (and enough of everything else...)

How to maximise your daily earnings?

The Linear Program

Linear Program for the Production Problem

$$\begin{array}{llcrcr}
\text{maximise} & x_1 & + & x_2 & & \\
\text{subject to} & & & & & \\
 & 4x_1 & + & x_2 & \leq & 20 \\
 & 2x_1 & + & x_2 & \leq & 10 \\
 & x_1 & + & 2x_2 & \leq & 14 \\
 & x_1, x_2 & & & \geq & 0
\end{array}$$

The solution of this linear program yields the optimal production schedule.

Formal Definition of Linear Program

- Given $a_1, a_2, \ldots, a_n$ and a set of variables $x_1, x_2, \ldots, x_n$, a linear function f is defined by

$$f(x_1, x_2, \ldots, x_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n.$$

- Linear Equality: $f(x_1, x_2, \ldots, x_n) = b$
- Linear Inequality: $f(x_1, x_2, \ldots, x_n) \gtreqless b$

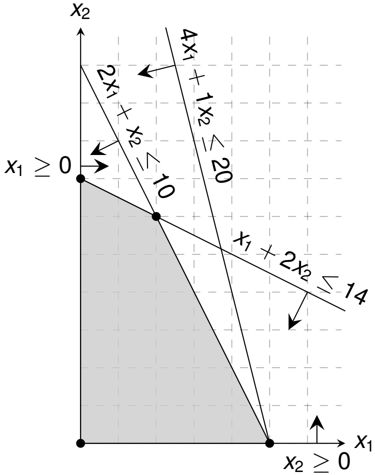
Linear Constraints

- Linear-Progamming Problem: either minimise or maximise a linear function subject to a set of linear constraints

Finding the Optimal Production Schedule

$$\begin{array}{llcl}
\text{maximise} & x_1 + x_2 & & \\
\text{subject to} & & & \\
 & 4x_1 + x_2 & \leq & 20 \\
 & 2x_1 + x_2 & \leq & 10 \\
 & x_1 + 2x_2 & \leq & 14 \\
 & x_1, x_2 & \geq & 0
\end{array}$$

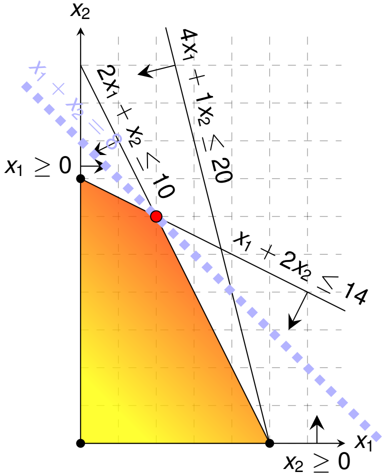
Any setting of x_1 and x_2 satisfying all constraints is a feasible solution



Question: Which aspect did we ignore in the formulation of the linear program?

Finding the Optimal Production Schedule

maximise	x_1	$+$	x_2		
subject to					
	$4x_1$	$+$	x_2	$\leq$	20
	$2x_1$	$+$	x_2	$\leq$	10
	x_1	$+$	$2x_2$	$\leq$	14
	x_1, x_2			$\geq$	0



Graphical Procedure: Move the line $x_1 + x_2 = z$ as far up as possible.

While the same approach also works for higher-dimensions, we need to take a more systematic and algebraic procedure.

Outline

Introduction

A Simple Example of a Linear Program

Formulating Problems as Linear Programs

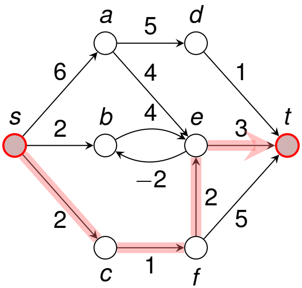
Standard and Slack Forms

Shortest Paths

Single-Pair Shortest Path Problem

- Given: directed graph $G = (V, E)$ with edge weights $w : E \to \mathbb{R}$, pair of vertices $s, t \in V$
- Goal: Find a path of minimum weight from s to t in G

$p = (v_0 = s, v_1, \ldots, v_k = t)$ such that $w(p) = \sum_{i=1}^{k} w(v_{k-1}, v_k)$ is minimised.



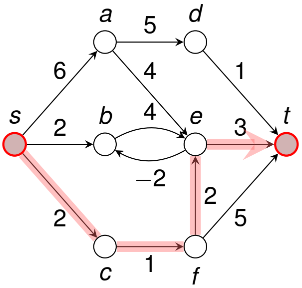
Exercise: Translate the SPSP problem into a linear program!

Shortest Paths

Single-Pair Shortest Path Problem

- Given: directed graph $G = (V, E)$ with edge weights $w : E \to \mathbb{R}$, pair of vertices $s, t \in V$
- Goal: Find a path of minimum weight from s to t in G

$p = (v_0 = s, v_1, \ldots, v_k = t)$ such that $w(p) = \sum_{i=1}^{k} w(v_{k-1}, v_k)$ is minimised.



Shortest Paths as LP

$$\begin{array}{llcll} \text{maximise} & d_t & & & \\ \text{subject to} & & & & \\ & d_v & \leq & d_u + w(u,v) & \text{for each edge } (u,v) \in E, \\ & d_s & = & 0. & \end{array}$$

Recall: When BELLMAN-FORD terminates, all these inequalities are satisfied.

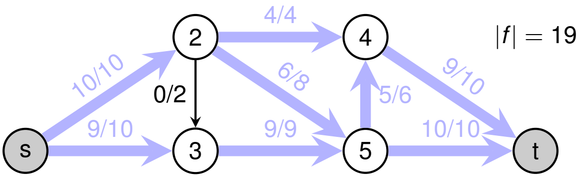
this is a maximisation problem!

Solution $\overline{d}$ satisfies $\overline{d}_v = \min_{u:\,(u,v)\in E}\left\{\overline{d}_u + w(u,v)\right\}$

Maximum Flow

Maximum Flow Problem

- Given: directed graph $G = (V, E)$ with edge capacities $c : E \to \mathbb{R}^+$ (recall $c(u, v) = 0$ if $(u, v) \notin E$), pair of vertices $s, t \in V$
- Goal: Find a maximum flow $f : V \times V \to \mathbb{R}$ from s to t which satisfies the capacity constraints and flow conservation



Maximum Flow as LP

$$
\begin{array}{lrcll}
\text{maximise} & \sum_{v\in V} f_{sv} & - & \sum_{v \in V} f_{vs} & \\
\text{subject to} & & & & \\
& f_{uv} & \leq & c(u,v) & \text{for each } u, v \in V, \\
& \sum_{v\in V} f_{vu} & = & \sum_{v\in V} f_{uv} & \text{for each } u \in V \setminus \{s,t\}, \\
& f_{uv} & \geq & 0 & \text{for each } u, v \in V.
\end{array}
$$

Minimum-Cost Flow

Extension of the Maximum Flow Problem

Minimum-Cost-Flow Problem

- Given: directed graph $G = (V, E)$ with capacities $c : E \to \mathbb{R}^+$, pair of vertices $s, t \in V$, cost function $a : E \to \mathbb{R}^+$, flow demand of d units
- Goal: Find a flow $f : V \times V \to \mathbb{R}$ from s to t with $|f| = d$ while minimising the total cost $\sum_{(u,v) \in E} a(u,v) f_{uv}$ incurrred by the flow.

Optimal Solution with total cost:
$\sum_{(u,v) \in E} a(u,v) f_{uv} = (2{\cdot}2)+(5{\cdot}2)+(3{\cdot}1)+(7{\cdot}1)+(1{\cdot}3) = 27$

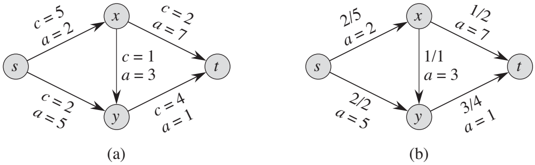


Figure 29.3 **(a)** An example of a minimum-cost-flow problem. We denote the capacities by c and the costs by a. Vertex s is the source and vertex t is the sink, and we wish to send 4 units of flow from s to t. **(b)** A solution to the minimum-cost flow problem in which 4 units of flow are sent from s to t. For each edge, the flow and capacity are written as flow/capacity.

Minimum Cost Flow as a LP

Minimum Cost Flow as LP

minimise $\sum_{(u,v)\in E} a(u,v) f_{uv}$

subject to

$$
\begin{aligned}
f_{uv} &\leq c(u,v) && \text{for } u,v \in V, \\
\sum_{v\in V} f_{vu} - \sum_{v \in V} f_{uv} &= 0 && \text{for } u \in V \setminus \{s,t\}, \\
\sum_{v\in V} f_{sv} - \sum_{v\in V} f_{vs} &= d\ , && \\
f_{uv} &\geq 0 && \text{for } u,v \in V.
\end{aligned}
$$

Real power of Linear Programming comes from the ability to solve **new problems**!

Outline

Introduction

A Simple Example of a Linear Program

Formulating Problems as Linear Programs

Standard and Slack Forms

Standard and Slack Forms

Standard Form

$$\text{maximise} \quad \sum_{j=1}^{n} c_j x_j$$

Objective Function

subject to

$$\left\{ \begin{array}{ll} \sum_{j=1}^{n} a_{ij} x_j \leq b_i & \text{for } i = 1, 2, \ldots, m \\ x_j \geq 0 & \text{for } j = 1, 2, \ldots, n \end{array} \right.$$

$n + m$ constraints

Non-Negativity Constraints

Standard Form (Matrix-Vector-Notation)

$$\text{maximise} \quad c^T x$$

Inner product of two vectors

subject to

$$Ax \leq b$$
$$x \geq 0$$

Matrix-vector product

Converting Linear Programs into Standard Form

Reasons for a LP not being in standard form:

1. The objective might be a minimisation rather than maximisation.
2. There might be variables without nonnegativity constraints.
3. There might be equality constraints.
4. There might be inequality constraints (with $\geq$ instead of $\leq$).

Goal: Convert linear program into an equivalent program which is in standard form

Equivalence: a correspondence (not necessarily a bijection) between solutions.

Converting into Standard Form (1/5)

Reasons for a LP not being in standard form:

1. The objective might be a minimisation rather than maximisation.

$$
\begin{array}{lrcrcl}
\text{minimise} & -2x_1 & + & 3x_2 & & \\
\text{subject to} & & & & & \\
 & x_1 & + & x_2 & = & 7 \\
 & x_1 & - & 2x_2 & \leq & 4 \\
 & x_1 & & & \geq & 0
\end{array}
$$

Negate objective function

$$
\begin{array}{lrcrcl}
\text{maximise} & 2x_1 & - & 3x_2 & & \\
\text{subject to} & & & & & \\
 & x_1 & + & x_2 & = & 7 \\
 & x_1 & - & 2x_2 & \leq & 4 \\
 & x_1 & & & \geq & 0
\end{array}
$$

Converting into Standard Form (2/5)

Reasons for a LP not being in standard form:

2. There might be variables without nonnegativity constraints.

$$
\begin{array}{llll}
\text{maximise} & 2x_1 - 3x_2 & & \\
\text{subject to} & & & \\
& x_1 + x_2 & = & 7 \\
& x_1 - 2x_2 & \leq & 4 \\
& x_1 & \geq & 0
\end{array}
$$

Replace x_2 by two non-negative variables x_2' and x_2''

$$
\begin{array}{llll}
\text{maximise} & 2x_1 - 3x_2' + 3x_2'' & & \\
\text{subject to} & & & \\
& x_1 + x_2' - x_2'' & = & 7 \\
& x_1 - 2x_2' + 2x_2'' & \leq & 4 \\
& x_1, x_2', x_2'' & \geq & 0
\end{array}
$$

Converting into Standard Form (3/5)

Reasons for a LP not being in standard form:

3. There might be equality constraints.

$$\begin{array}{ll} \text{maximise} & 2x_1 - 3x_2' + 3x_2'' \\ \text{subject to} & \\ & x_1 + x_2' - x_2'' = 7 \\ & x_1 - 2x_2' + 2x_2'' \leq 4 \\ & x_1, x_2', x_2'' \geq 0 \end{array}$$

Replace each equality by two inequalities.

$$\begin{array}{ll} \text{maximise} & 2x_1 - 3x_2' + 3x_2'' \\ \text{subject to} & \\ & x_1 + x_2' - x_2'' \leq 7 \\ & x_1 + x_2' - x_2'' \geq 7 \\ & x_1 - 2x_2' + 2x_2'' \leq 4 \\ & x_1, x_2', x_2'' \geq 0 \end{array}$$

Converting into Standard Form (4/5)

Reasons for a LP not being in standard form:

4. There might be inequality constraints (with $\geq$ instead of $\leq$).

$$
\begin{array}{llllllll}
\text{maximise} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
& x_1 & + & x_2' & - & x_2'' & \geq & 7 \\
& x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
& & x_1, x_2', x_2'' & & & & \geq & 0
\end{array}
$$

Negate respective inequalities.

$$
\begin{array}{llllllll}
\text{maximise} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
& -x_1 & - & x_2' & + & x_2'' & \leq & -7 \\
& x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
& & x_1, x_2', x_2'' & & & & \geq & 0
\end{array}
$$

Converting into Standard Form (5/5)

Rename variable names (for consistency).

$$\begin{array}{lrcrcrcr}
\text{maximise} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & - & x_3 & \leq & 7 \\
& -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
& x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
& x_1, x_2, x_3 & & & & & \geq & 0
\end{array}$$

It is always possible to convert a linear program into standard form.

Converting Standard Form into Slack Form (1/3)

Goal: Convert standard form into slack form, where all constraints except for the non-negativity constraints are equalities.

For the simplex algorithm, it is more convenient to work with equality constraints.

Introducing Slack Variables

- Let $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ be an inequality constraint
- Introduce a slack variable s by

$$s = b_i - \sum_{j=1}^{n} a_{ij} x_j$$

$$s \geq 0.$$

s measures the slack between the two sides of the inequality.

- Denote slack variable of the i-th inequality by x_{n+i}

Converting Standard Form into Slack Form (2/3)

$$\begin{array}{lrcrcrcr}
\text{maximise} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & - & x_3 & \leq & 7 \\
& -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
& x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
& & & x_1, x_2, x_3 & & & \geq & 0
\end{array}$$

Introduce slack variables

$$\begin{array}{lrcrcrcrcr}
\text{maximise} & 2x_1 & - & 3x_2 & + & 3x_3 & & & & \\
\text{subject to} & & & & & & & & & \\
& x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
& x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
& x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
& & & x_1, x_2, x_3, x_4, x_5, x_6 & & \geq & & 0 & &
\end{array}$$

Converting Standard Form into Slack Form (3/3)

$$\begin{array}{llllllllll}
\text{maximise} & 2x_1 & - & 3x_2 & + & 3x_3 \\
\text{subject to} \\
 & x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
 & x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
 & x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
 & \multicolumn{5}{c}{x_1, x_2, x_3, x_4, x_5, x_6} & \geq & 0
\end{array}$$

Use variable z to denote objective function and omit the nonnegativity constraints.

$$\begin{array}{lllllllll}
z & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}$$

This is called slack form.

Basic and Non-Basic Variables

$$
\begin{array}{rcrcrcrcr}
z & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}
$$

Basic Variables: $B = \{4, 5, 6\}$

Non-Basic Variables: $N = \{1, 2, 3\}$

Slack Form (Formal Definition)

Slack form is given by a tuple (N, B, A, b, c, v) so that

$$z = v + \sum_{j \in N} c_j x_j$$

$$x_i = b_i - \sum_{j \in N} a_{ij} x_j \qquad \text{for } i \in B,$$

and all variables are non-negative.

Variables/Coefficients on the right hand side are indexed by B and N.

Slack Form (Example)

$$
\begin{aligned}
z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\
x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2}
\end{aligned}
$$

Slack Form Notation

- $B = \{1, 2, 4\}$, $N = \{3, 5, 6\}$
- $$A = \begin{pmatrix} a_{13} & a_{15} & a_{16} \\ a_{23} & a_{25} & a_{26} \\ a_{43} & a_{45} & a_{46} \end{pmatrix} = \begin{pmatrix} -1/6 & -1/6 & 1/3 \\ 8/3 & 2/3 & -1/3 \\ 1/2 & -1/2 & 0 \end{pmatrix}$$
- $$b = \begin{pmatrix} b_1 \\ b_2 \\ b_4 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 18 \end{pmatrix}, \quad c = \begin{pmatrix} c_3 \\ c_5 \\ c_6 \end{pmatrix} = \begin{pmatrix} -1/6 \\ -1/6 \\ -2/3 \end{pmatrix}$$
- $v = 28$