Ordered pairing

Notation:

$$(a, b) \quad \text{or} \quad \langle a, b \rangle$$

Fundamental property:

$$(a, b) = (x, y) \implies a = x \wedge b = y$$

A construction:

For every pair a and b,

$$\langle a, b \rangle = \{ \{ a \}, \{ a, b \} \}$$

defines an *ordered pairing* of a and b.

Proposition 108 (Fundamental property of ordered pairing)
For all a, b, x, y,

$$\langle a, b\rangle = \langle x, y\rangle \iff \big(a = x \wedge b = y\big) \quad .$$

PROOF:

Products

The *product* $A \times B$ of two sets A and B is the set

$$A \times B = \{\, x \mid \exists\, a \in A, b \in B.\, x = (a, b)\,\}$$

where

$$\begin{array}{l} \forall a_1, a_2 \in A, b_1, b_2 \in B. \\ \quad (a_1, b_1) = (a_2, b_2) \iff (a_1 = a_2 \ \wedge \ b_1 = b_2) \end{array} \quad .$$

Thus,

$$\forall\, x \in A \times B.\, \exists!\, a \in A.\, \exists!\, b \in B.\, x = (a, b) \quad .$$

Pattern-matching notation

Example: The subset of ordered pairs from a set A with equal components is formally

$$\{ x \in A \times A \mid \exists a_1 \in A.\, \exists a_2 \in A.\, x = (a_1, a_2) \ \wedge \ a_1 = a_2 \}$$

but often abbreviated using *pattern-matching notation* as

$$\{ (a_1, a_2) \in A \times A \mid a_1 = a_2 \} \ .$$

Notation: For a property $P(a, b)$ with a ranging over a set A and b ranging over a set B,

$$\{ (a, b) \in A \times B \mid P(a, b) \}$$

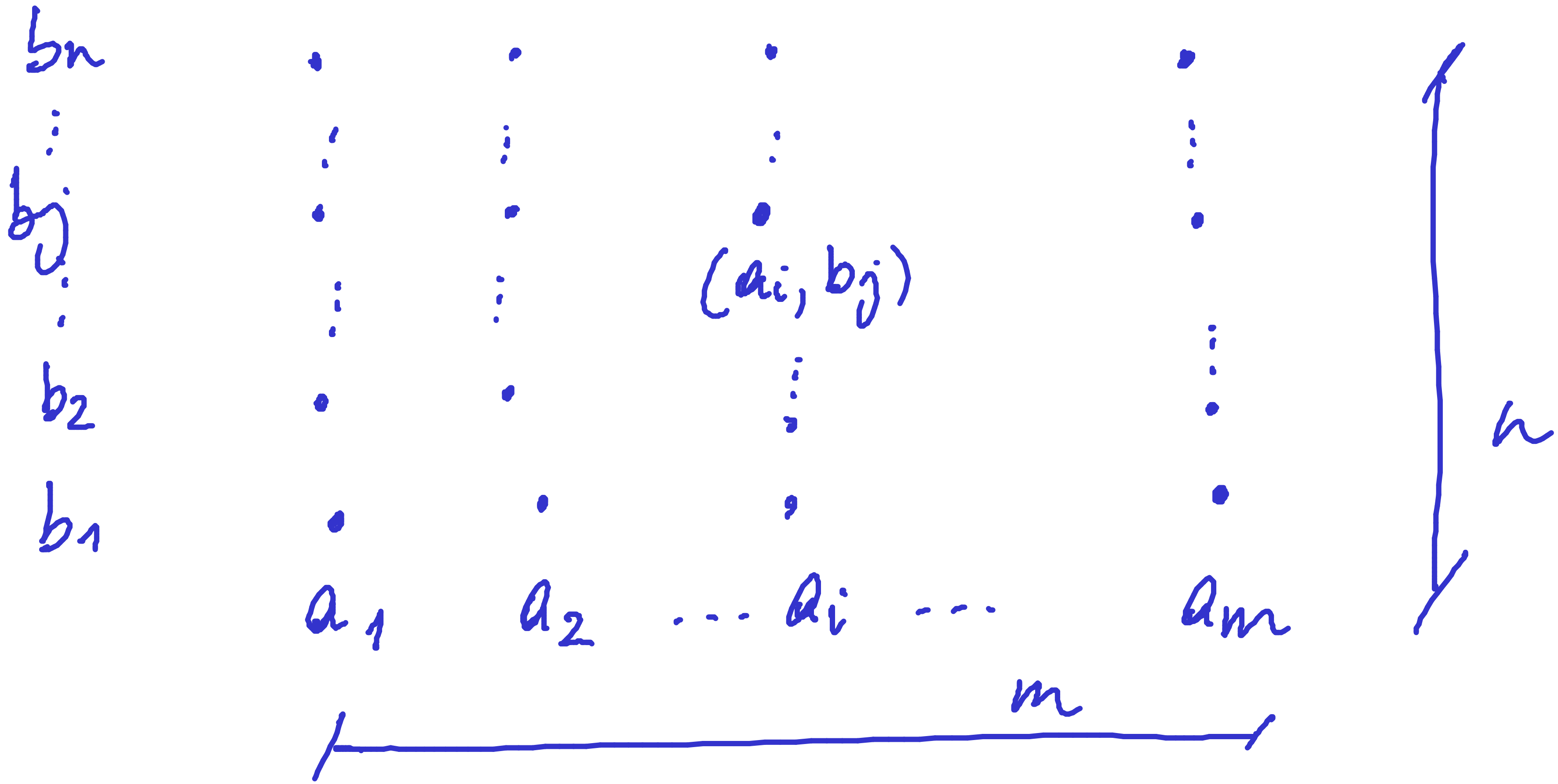
abbreviates

$$\{ x \in A \times B \mid \exists a \in A.\, \exists b \in B.\, x = (a, b) \ \wedge \ P(a, b) \} \ .$$

Proposition 110 *For all finite sets* A *and* B,

$$\#(A \times B) = \#A \cdot \#B \quad .$$

PROOF IDEA: $A = \{a_1, \ldots, a_m\}$ $B = \{b_1, \ldots, b_n\}$



Sets and logic

$\mathcal{P}(\mathcal{U})$	$\{\,\textbf{false}\,,\ \textbf{true}\,\}$
$\emptyset$	**false**
$\mathcal{U}$	**true**
$\cup$	$\vee$
$\cap$	$\wedge$
$(\cdot)^{\mathrm{c}}$	$\neg(\cdot)$
$\bigcup$	$\exists$
$\bigcap$	$\forall$

Big unions

Example:

$[5] = \{0, 1, 2, 3, 4\}$.

- Consider the family of sets

$$\mathcal{T} = \left\{ T \subseteq [5] \;\middle|\; \begin{array}{l} \text{the sum of the elements of} \\ T \text{ is less than or equal } 2 \end{array} \right\}$$

$$= \{\emptyset, \{0\}, \{1\}, \{0,1\}, \{0,2\}, \{2\}\}$$

- The *big union* of the family $\mathcal{T}$ is the set $\bigcup \mathcal{T}$ given by the union of the sets in $\mathcal{T}$:

$$n \in \bigcup \mathcal{T} \iff \exists T \in \mathcal{T}. n \in T \quad .$$

Hence, $\bigcup \mathcal{T} = \{0, 1, 2\}$.

$\underline{\underline{NB}}$: $\mathcal{F} \in \mathcal{P}(\mathcal{P}(U)) \Rightarrow \bigcup \mathcal{F} \in \mathcal{P}(U)$

Definition 111 *Let* U *be a set. For a collection of sets* $\mathcal{F} \in \mathcal{P}(\mathcal{P}(U))$, *we let the* big union *(relative to* U*) be defined as*

$$\bigcup \mathcal{F} = \{ x \in U \mid \exists A \in \mathcal{F}. x \in A \} \in \mathcal{P}(U) \quad .$$

$\bigcup(\bigcup \mathcal{F}) \in \mathcal{P}(\mathcal{U})$

$\Uparrow$

$\bigcup \mathcal{F} \in \mathcal{P}(\mathcal{P}(\mathcal{U}))$

$\Uparrow$

Proposition 112 *For all* $\mathcal{F} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{U})))$,

$$\bigcup\left(\bigcup \mathcal{F}\right) = \overbrace{\bigcup\left\{ \bigcup \mathcal{A} \in \mathcal{P}(\mathbb{U}) \;\middle|\; \mathcal{A} \in \mathcal{F} \right\}}^{(*)} \in \mathcal{P}(\mathbb{U}) \quad .$$

PROOF:

$$\begin{aligned}
& x \in \bigcup(\bigcup \mathcal{F}) \\
\iff\ & \exists X.\ X \in \bigcup \mathcal{F} \wedge x \in X \\
\iff\ & \exists X.\ \exists A.\ A \in \mathcal{F} \wedge X \in A \wedge x \in X. \\
\iff\ & \exists A.\ A \in \mathcal{F} \wedge \exists X.\ X \in A \wedge x \in X \\
\iff\ & \exists A.\ A \in \mathcal{F} \wedge x \in \bigcup A \iff x \in (*)
\end{aligned}$$

$\boxtimes$

Big intersections

Example:

- Consider the family of sets

$$\mathcal{S} = \left\{ S \subseteq [5] \;\middle|\; \text{the sum of the elements of } S \text{ is } 6 \right\}$$

$$= \{\{2,4\},\{0,2,4\},\{1,2,3\},\{0,1,2,3\}\}$$

- The *big intersection* of the family $\mathcal{S}$ is the set $\bigcap \mathcal{S}$ given by the intersection of the sets in $\mathcal{S}$:

$$n \in \bigcap \mathcal{S} \iff \forall S \in \mathcal{S}.\, n \in S \quad .$$

Hence, $\bigcap \mathcal{S} = \{2\}$.

Definition 113 *Let $\mathcal{U}$ be a set. For a collection of sets $\mathcal{F} \subseteq \mathcal{P}(\mathcal{U})$, we let the* <u>big intersection</u> *(relative to $\mathcal{U}$) be defined as*

$$\bigcap \mathcal{F} \;=\; \big\{\, x \in \mathcal{U} \mid \forall A \in \mathcal{F}.\, x \in A \,\big\} \quad .$$

Theorem 114 *Let*

$$\mathcal{F} = \left\{ S \subseteq \mathbb{R} \;\middle|\; (0 \in S) \wedge \left(\forall x \in \mathbb{R}.\, x \in S \implies (x+1) \in S\right) \right\} .$$

Then, (i) $\mathbb{N} \in \mathcal{F}$ *and* (ii) $\mathbb{N} \subseteq \bigcap \mathcal{F}$. *Hence,* $\bigcap \mathcal{F} = \mathbb{N}$.

PROOF:

$\mathbb{R} \in \mathcal{F}$

$\mathbb{Q} \in \mathcal{F}$

$\mathbb{Z} \in \mathcal{F}$

$\mathbb{N} \in \mathcal{F}$

$\boxed{\forall x \in \mathbb{N}.\, \forall S \in \mathcal{F}.\, x \in S}$

$\mathbb{N} \subseteq \bigcap \mathcal{F}$

$\iff \forall x.\; x \in \mathbb{N} \Rightarrow x \in \bigcap \mathcal{F}$

Let $x \in \mathbb{N}$.

RTP: $x \in \bigcap \mathcal{F}$

$\iff \forall S \in \mathcal{F}.\; x \in S$.

Proposition 115 *Let* $\mathcal{U}$ *be a set and let* $\mathcal{F} \subseteq \mathcal{P}(\mathcal{U})$ *be a family of subsets of* $\mathcal{U}$.

To show S is $\bigcup \mathcal{F}$ establish:

1. *For all* $S \in \mathcal{P}(\mathcal{U})$,

$$S = \bigcup \mathcal{F}$$

iff

$$\textcircled{1}\ \big[\forall A \in \mathcal{F}.\, A \subseteq S\big]$$

$$\wedge\ \textcircled{2}\ \big[\forall X \in \mathcal{P}(\mathcal{U}).\, (\forall A \in \mathcal{F}.\, A \subseteq X) \Rightarrow S \subseteq X\big]$$

To show T is $\bigcap \mathcal{F}$ establish:

2. *For all* $T \in \mathcal{P}(\mathcal{U})$,

$$T = \bigcap \mathcal{F}$$

iff

$$\textcircled{1}\ \big[\forall A \in \mathcal{F}.\, T \subseteq A\big]$$

$$\wedge\ \textcircled{2}\ \big[\forall Y \in \mathcal{P}(\mathcal{U}).\, (\forall A \in \mathcal{F}.\, Y \subseteq A) \Rightarrow Y \subseteq T\big]$$

Union axiom

$$\boxed{\text{Every collection of sets has a union.}}$$

$$\bigcup \mathcal{F}$$

$$x \in \bigcup \mathcal{F} \iff \exists X \in \mathcal{F}.\, x \in X$$

For *non-empty* $\mathcal{F}$ we also have

$$\bigcap \mathcal{F}$$

defined by

$$\forall x.\ x \in \bigcap \mathcal{F} \iff \big(\forall X \in \mathcal{F}.\, x \in X\big) \quad .$$

S a set and t a element consider

$$\{t\} \times S = \{ (t, s) \mid s \in S \}$$

Disjoint unions

Definition 116 *The* disjoint union $A \uplus B$ *of two sets* A *and* B *is the set*

$$A \uplus B \; = \; \big(\{1\} \times A\big) \cup \big(\{2\} \times B\big) \quad .$$

Thus,

$$(\{1\} \times A) \cap (\{2\} \times B) = \emptyset$$

$$\forall x.\, x \in (A \uplus B) \iff \big(\, \exists a \in A.\, x = (1, a)\,\big) \vee \big(\, \exists b \in B.\, x = (2, b)\,\big) \,.$$

datatype

(α, β) disjointunion = left of α | right of β.

Proposition 118 *For all finite sets* A *and* B,

$$A \cap B = \emptyset \implies \#(A \cup B) = \#A + \#B \quad .$$

PROOF IDEA:

Let $A \cap B = \emptyset$ $\quad A = \{a_1 \ldots a_m\}$ $B = \{b_1 \ldots b_n\}$

Then $A \cup B = \{a_1 \ldots a_m\, b_1 \ldots b_n\}$

$\# A \cup B = m + n$.

Corollary 119 *For all finite sets* A *and* B,

$$\#(A \uplus B) = \#A + \#B \quad .$$