Slides
for Part IA CST 2023/24

# Discrete Mathematics

<www.cl.cam.ac.uk/teaching/2324/DiscMath>

**Prof Marcelo Fiore**
Marcelo.Fiore@cl.cam.ac.uk

## What are we up to ?

- Learn to read and write, and also work with, mathematical arguments.
- Doing some basic discrete mathematics.
- Getting a taste of computer science applications.

## Lecture plan

I. Proofs.

II. Numbers.

III. Sets.

IV. Regular languages and finite automata.

# Proofs

## Objectives

- To develop techniques for analysing and understanding mathematical statements.
- To be able to present logical arguments that establish mathematical statements in the form of clear proofs.
- To prove Fermat's Little Theorem, a basic result in the theory of numbers that has many applications in computer science.

# Proofs in practice

We are interested in examining the following statement:

The product of two odd integers is odd.

This seems innocuous enough, but it is in fact full of baggage.
For instance, it presupposes that you know:

- what a statement is;
- what the integers $(\ldots, -1, 0, 1, \ldots)$ are, and that amongst them there is a class of odd ones $(\ldots, -3, -1, 1, 3, \ldots)$;
- what the product of two integers is, and that this is in turn an integer.

More precisely put, we may write:

If $m$ and $n$ are odd integers then so is $m \cdot n$.

which further presupposes that you know:

- what variables are;
- what

  if ... then ...

  statements are, and how one goes about proving them;
- that the symbol "$\cdot$" is commonly used to denote the product operation.

Even more precisely, we should write

> For all integers $m$ and $n$, if $m$ and $n$ are odd then so is $m \cdot n$.

which now additionally presupposes that you know:

- what

  for all . . .

  statements are, and how one goes about proving them.

Thus, in trying to understand and then prove the above statement, we are assuming quite a lot of *mathematical jargon* that one needs to learn and practice with to make it a useful, and in fact very powerful, tool.

# Some mathematical jargon

**Statement**

A sentence that is either true or false — but not both.

**Example 1**

$$\text{‘}e^{i\pi} + 1 = 0\text{’}$$

**Non-example**

‘This statement is false’

**Predicate**

A statement whose truth depends on the value of one or more variables.

**Example 2**

*1.* '$e^{i\,x} = \cos x + i \sin x$'

*2.* *'the function $f$ is differentiable'*

**Theorem**

A very important true statement.

**Proposition**

A less important but nonetheless interesting true statement.

**Lemma**

A true statement used in proving other true statements.

**Corollary**

A true statement that is a simple deduction from a theorem or proposition.

**Example 3**

1. *Fermat's Last Theorem*
2. *The Pumping Lemma*

**Conjecture**

A statement believed to be true, but for which we have no proof.

**Example 4**

1. *Goldbach's Conjecture*
2. *The Riemann Hypothesis*

**Proof**

Logical explanation of why a statement is true; a method for establishing truth.

**Logic**

The study of methods and principles used to distinguish good (correct) from bad (incorrect) reasoning.

**Example 5**

1. *Classical predicate logic*
2. *Hoare logic*
3. *Temporal logic*

**Axiom**

A basic assumption about a mathematical situation.

Axioms can be considered facts that do not need to be proved (just to get us going in a subject) or they can be used in definitions.

**Example 6**

1. *Euclidean Geometry*
2. *Riemannian Geometry*
3. *Hyperbolic Geometry*

**Definition**

An explanation of the mathematical meaning of a word (or phrase).

The word (or phrase) is generally defined in terms of properties.

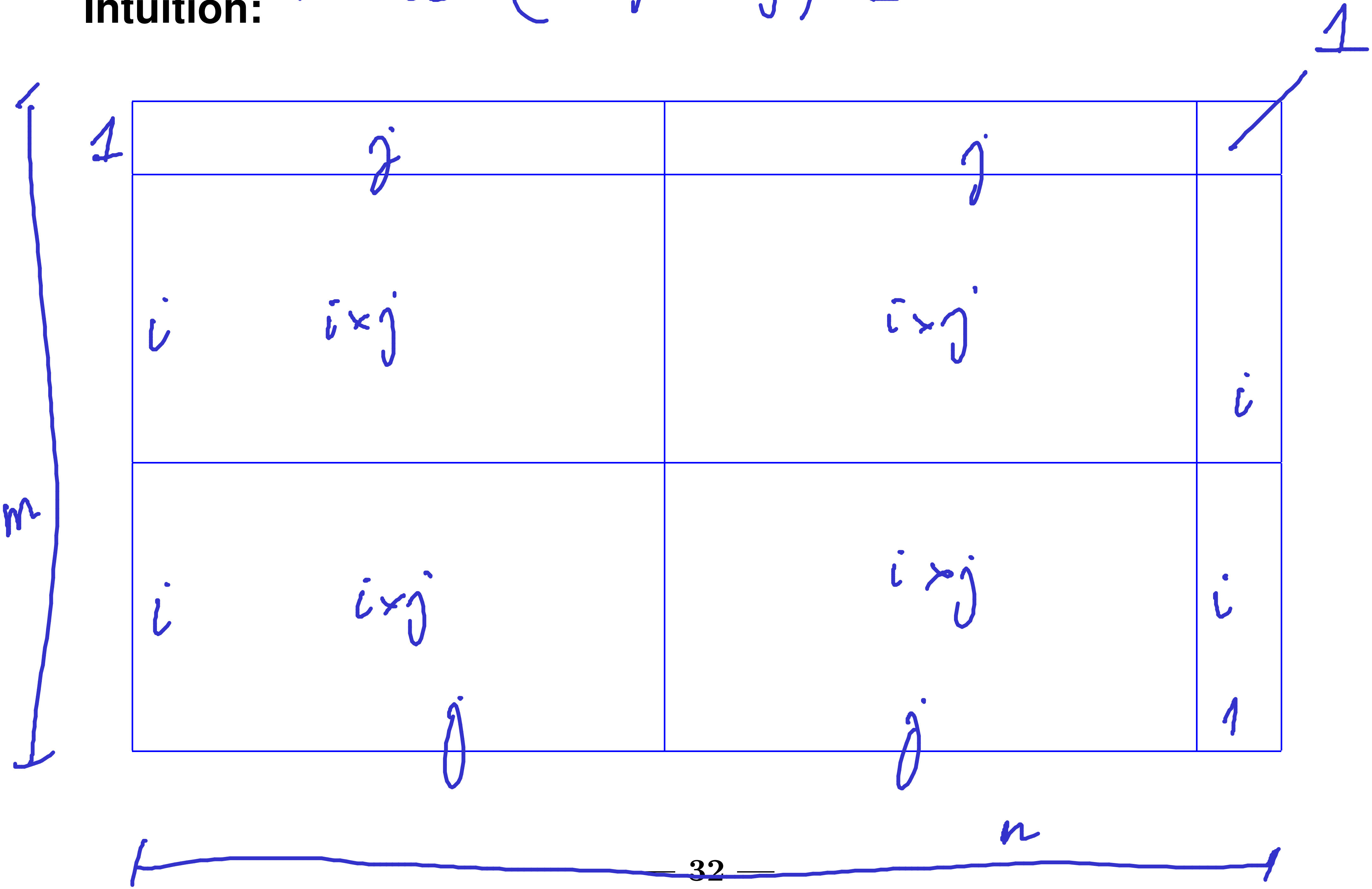
**Warning:** It is vitally important that you can recall definitions precisely. A common problem is not to be able to advance in some problem because the definition of a word is unknown.

# Definition, theorem, intuition, proof in practice

**Definition 7** *An integer is said to be* odd *whenever it is of the form* $2 \cdot i + 1$ *for some (necessarily unique) integer* $i$.

**Proposition 8** *For all integers* $m$ *and* $n$*, if* $m$ *and* $n$ *are odd then so is* $m \cdot n$.

**Intuition:** $m \times n = 2(2 i \times j + i + j) + 1$



PROOF OF Proposition 8:

Let $m$ and $n$ be integers.

As $m$ is odd, it is of the form $2i+1$, for some integer $i$. Analogously, $n$ is of the form $2j+1$ for some integer $j$. Therefore

$$m \times n = (2i+1) \times (2j+1)$$

$$= 2(2i \times j + i + j) + 1$$ by alg. manipulation.

That is, $m \times n$ is odd.

$\boxtimes$

# Simple and composite statements

A statement is *simple* (or *atomic*) when it cannot be broken into other statements, and it is *composite* when it is built by using several (simple or composite statements) connected by *logical* expressions (e.g., if. . . then. . . ; . . . implies . . . ; . . . if and only if . . . ; . . . and. . . ; either . . . or . . . ; it is not the case that . . . ; for all . . . ; there exists . . . ; etc.)

**Examples:**

'$2$ is a prime number'

'for all integers $m$ and $n$, if $m \cdot n$ is even then either $n$ or $m$ are even'

# Proof Structure

| Assumptions | Goals |
| --- | --- |
| statements that may be used for deduction | statements to be established |

# Implication

Theorems can usually be written in the form

**if** a collection of *assumptions* holds,
**then** so does some *conclusion*

or, in other words,

a collection of *assumptions* **implies** some *conclusion*

or, in symbols,

a collection of *hypotheses* $\Longrightarrow$ some *conclusion*

**NB** Identifying precisely what the assumptions and conclusions are is the first goal in dealing with a theorem.

Assumptions | Goals.

$A \Rightarrow B$ | $P \Rightarrow Q$ ?

## Implications

- How to *prove* them as goals.
- How to *use* them as assumptions.

# How to prove implication goals

**The main proof strategy for implication:**

To prove a goal of the form

$$P \implies Q$$

assume that $P$ is true and prove $Q$.

**NB** *Assuming* is not *asserting*! Assuming a statement amounts to the same thing as adding it to your list of hypotheses.

**Proof pattern:**

In order to prove that

$$P \implies Q$$

1. Write: Assume $P$.
2. Show that $Q$ logically follows.

**Scratch work:**

Before using the strategy

Assumptions | Goal
--- | ---
| $P \implies Q$
$\vdots$ |

After using the strategy

Assumptions | Goal
--- | ---
| $Q$
$\vdots$ |
$P$ |

**Proposition 8** *If $m$ and $n$ are odd integers, then so is $m \cdot n$.*

PROOF:

To show

(m and n odd integers) $\Rightarrow$ (m·n odd integers)

Assume

m and n are odd integers.

RTP (Required To Prove):

m·n is odd integers.

...

**Definition 9** *A real number is:*

- rational *if it is of the form* $m/n$ *for a pair of integers* $m$ *and* $n$*; otherwise it is* irrational*.*
- positive *if it is greater than* $0$*, and* negative *if it is smaller than* $0$*.*
- nonnegative *if it is greater than or equal* $0$*, and* nonpositive *if it is smaller than or equal* $0$*.*
- natural *if it is a nonnegative integer.*

**Proposition 10** *Let $x$ be a positive real number. If $\sqrt{x}$ is rational then so is $x$.*

PROOF: Let $x$ be a positive real number.

Show

$\sqrt{x}$ rational $\Rightarrow$ $x$ rational.

Assume $\sqrt{x}$ rational (*)

RTP $x$ rational.

By assumption (*), $\sqrt{x} = m/n$ for integers $m$ and $n$.

We have $x = (\sqrt{x})^2 = (m/n)^2 = m^2/n^2$ and therefore $x$ is rational. ☒

# How to use implication assumptions

## Logical Deduction by Modus Ponens

A main rule of *logical deduction* is that of *Modus Ponens*:

> From the statements $P$ and $P \implies Q$,
> the statement $Q$ follows.

or, in other words,

> If $P$ and $P \implies Q$ hold then so does $Q$.

or, in symbols,

$$\frac{P \qquad P \implies Q}{Q}$$

**The use of implications:**

To use an assumption of the form $P \implies Q$, aim at establishing $P$.
Once this is done, by Modus Ponens, one can conclude $Q$ and so further assume it.

**Theorem 11** *Let* $P_1$, $P_2$, *and* $P_3$ *be statements. If* $P_1 \implies P_2$ *and* $P_2 \implies P_3$ *then* $P_1 \implies P_3$.

PROOF: Let $P_1, P_2, P_3$ be statements.

<u>Assume</u>: (1) $P_1 \Rightarrow P_2$ and (2) $P_2 \Rightarrow P_3$.

<u>Goal</u>: $P_1 \Rightarrow P_3$

<u>Assume</u>: (3) $P_1$

<u>Goal</u>: $P_3$

From (1) and (3), we have (4) $P_2$ and from (2) we have $P_3$ as required. ☒

# Bi-implication

Some theorems can be written in the form

P is equivalent to Q

or, in other words,

P implies Q, and vice versa

or

Q implies P, and vice versa

or

P if, and only if, Q

P iff Q

or, in symbols,

$$P \iff Q$$