Denotational Semantics

Meven Lennon-Bertrand

Lectures for Part II CST 2023/2024

Practicalities

- My mail: mgapb2@cam.ac.uk. Do not hesitate to ask questions!
- Course notes will be updated, keep an eye on the course webpage.

Introduction

What is this course about?

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What is this course about?

- Formal methods: tools for the specification, development, analysis and verification of software and hardware systems.
- Programming language theory: how to design, implement and reason about programming languages?
- Programming language semantics: what is the (mathematical) meaning of a program?

Goal: give an **abstract** and **compositional** (mathematical) model of programs.

Why should we care?

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- **Insight**: exposes the mathematical "essence" of programming language concepts.
- **Language design**: feedback from semantic concepts (monads, algebraic effects & effect handlers...).
- **Rigour**: semantics is necessary to specify/justify formal methods (compilers, type systems, code analysis, certification...).

- Operational
- Axiomatic
- Denotational

- **Operational**: meaning of a program in terms of the *steps of computation* it takes during execution (see Part IB Semantics).
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- Denotational

- **Operational**: meaning of a program in terms of the *steps of computation* it takes during execution (see Part IB Semantics).
- **Axiomatic**: indirect meaning of a program in terms of a *program logic* to reason about its properties (see Part II Hoare Logic & Model Checking).
- **Denotational**: meaning of a program defined abstractly as object of some suitable *mathematical structure* (see this course).

Denotational semantics in a nutshell

$$\begin{array}{rcl} \text{Syntax} & \xrightarrow{[\![-]\!]} & \text{Semantics} \\ \text{Program } P & \mapsto & \text{Denotation } [\![P]\!] \\ \\ \text{Recursive program} & \mapsto & \text{Partial recursive function} \\ \text{Boolean circuit} & \mapsto & \text{Boolean function} \\ & \dots & \end{array}$$

Denotational semantics in a nutshell

$$
\begin{array}{rcl}
\text{Syntax} & \xrightarrow{[\![-]\!]} & \text{Semantics} \\
\text{Program } P & \mapsto & \text{Denotation } [\![P]\!] \\
\\
\text{Recursive program} & \mapsto & \text{Partial recursive function} \\
\text{Boolean circuit} & \mapsto & \text{Boolean function} \\
& \cdots & \\
\text{Type} & \mapsto & \text{Domain} \\
\text{Program} & \mapsto & \text{Continuous functions between domains}
\end{array}
$$

Properties of denotational semantics

Abstraction

- mathematical object, implementation/machine independent;
- captures the abstract essence of programming language concepts;
- should relate to practical implementations, though...

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Compositionality

- The denotation of a phrase is defined using the *denotation* of its sub-phrases.
- $[\![P]\!]$ represents the contribution of P to *any* program containing P.
- Much more flexible than whole-program semantics.

Introduction

A basic example

Commands

$$C \in \mathbf{Comm} ::= \texttt{skip} \mid L := A \mid C; C \mid \texttt{if}\ B\ \texttt{then}\ C\ \texttt{else}\ C \mid \texttt{while}\ B\ \texttt{do}\ C$$

IMP syntax

Commands

ranges over a set $\mathbb{L}$ of *locations*

$$C \in \mathbf{Comm} ::= \texttt{skip} \mid L := A \mid C; C \mid \texttt{if}\ B\ \texttt{then}\ C\ \texttt{else}\ C \mid \texttt{while}\ B\ \texttt{do}\ C$$

Arithmetic expressions

$$A \in \mathbf{Aexp} ::= \underline{n} \mid L \mid A + A \mid \ldots$$

Commands

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ranges over *integers* →

Arithmetic expressions

$$A \in \mathbf{Aexp} ::= \underline{n} \mid L \mid A + A \mid \dots$$

Commands

$$C \in \mathbf{Comm} ::= \texttt{skip} \mid L := A \mid C; C \mid \texttt{if}\ B\ \texttt{then}\ C\ \texttt{else}\ C \mid \texttt{while}\ B\ \texttt{do}\ C$$

Arithmetic expressions

$$A \in \mathbf{Aexp} ::= \underline{n} \mid L \mid A + A \mid \ldots$$

Boolean expressions

$$B \in \mathbf{Bexp} ::= \texttt{true} \mid \texttt{false} \mid A = A \mid \neg B \mid \ldots$$

Commands

$$C \in \mathbf{Comm} ::= \texttt{skip} \mid L := A \mid C; C \mid \texttt{if}\ B\ \texttt{then}\ C\ \texttt{else}\ C \mid \texttt{while}\ B\ \texttt{do}\ C$$

$$\mathcal{A}: \quad \mathbf{Aexp} \to \mathbb{Z}$$

where

$$\mathbb{Z} \quad = \quad \{\ldots, -1, 0, 1, \ldots\}$$

$$\begin{aligned} \mathcal{A} &: \quad \mathbf{Aexp} \to \mathbb{Z} \\ \mathcal{B} &: \quad \mathbf{Bexp} \to \mathbb{B} \end{aligned}$$

where

$$\begin{aligned} \mathbb{Z} &= \{\ldots, -1, 0, 1, \ldots\} \\ \mathbb{B} &= \{\text{true}, \text{false}\} \end{aligned}$$

Arithmetic expressions?

$$\mathcal{A}[\![\underline{n}]\!] = n$$

$$\mathcal{A}[\![A_1 + A_2]\!] = \mathcal{A}[\![A_1]\!] + \mathcal{A}[\![A_2]\!]$$

Arithmetic expressions?

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$$\mathcal{A}[\![A_1 + A_2]\!] = \mathcal{A}[\![A_1]\!] + \mathcal{A}[\![A_2]\!]$$

$$\mathcal{A}[\![L]\!] = ???$$

$$\text{State} = (\mathbb{L} \to \mathbb{Z})$$

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$$\mathcal{A} : \mathbf{Aexp} \to (\text{State} \to \mathbb{Z})$$
$$\mathcal{B} : \mathbf{Bexp} \to (\text{State} \to \mathbb{B})$$

where

$$\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$$
$$\mathbb{B} = \{\text{true}, \text{false}\}\,.$$

$$\text{State} = (\mathbb{L} \to \mathbb{Z})$$

$$\begin{aligned}
\mathcal{A} &: \mathbf{Aexp} \to (\text{State} \to \mathbb{Z}) \\
\mathcal{B} &: \mathbf{Bexp} \to (\text{State} \to \mathbb{B}) \\
\mathcal{C} &: \mathbf{Comm} \to (\text{State} \rightharpoonup \text{State})
\end{aligned}$$

where $\rightharpoonup$ denotes partial functions and

$$\begin{aligned}
\mathbb{Z} &= \{\ldots, -1, 0, 1, \ldots\} \\
\mathbb{B} &= \{\text{true}, \text{false}\}\,.
\end{aligned}$$

Semantics of arithmetic expressions

$$
\begin{aligned}
\mathcal{A}\llbracket \underline{n} \rrbracket &= \lambda s \in \text{State}.\ n \\
\mathcal{A}\llbracket A_1 + A_2 \rrbracket &= \lambda s \in \text{State}.\ \mathcal{A}\llbracket A_1 \rrbracket (s) + \mathcal{A}\llbracket A_2 \rrbracket (s)
\end{aligned}
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Semantics of arithmetic expressions

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\mathcal{A}\left[\!\left[\underline{n}\right]\!\right] &= \lambda s \in \text{State}.\ n \\
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\mathcal{A}[\![L]\!] &= \lambda s \in \text{State}.\ s(L)
\end{aligned}
$$

Semantics of boolean expressions

$$
\begin{array}{rcl}
\mathcal{B}[\![\texttt{true}]\!] & = & \lambda s \in \mathrm{State}.\ \mathrm{true} \\
\mathcal{B}[\![\texttt{false}]\!] & = & \lambda s \in \mathrm{State}.\ \mathrm{false} \\
\mathcal{B}[\![A_1 = A_2]\!] & = & \lambda s \in \mathrm{State}.\ \mathrm{eq}\left(\mathcal{A}[\![A_1]\!]\,(s), \mathcal{A}[\![A_2]\!]\,(s)\right) \\
 & & \text{where } \mathrm{eq}(a, a') = \begin{cases} \mathrm{true} & \text{if } a = a' \\ \mathrm{false} & \text{if } a \neq a' \end{cases}
\end{array}
$$

Semantics of commands

$$\mathcal{C}[\![\texttt{skip}]\!] \quad = \quad \lambda s \in \text{State}.\ s$$

$$\mathcal{C}[\![\texttt{skip}]\!] = \lambda s \in \mathrm{State}.\, s$$

$$\mathcal{C}[\![\texttt{if } B \texttt{ then } C \texttt{ else } C']\!] = \lambda s \in \mathrm{State}.\, \mathrm{if}\,(\mathcal{C}[\![B]\!](s), \mathcal{C}[\![C]\!](s), \mathcal{C}[\![C']\!](s))$$

$$\text{where } \mathrm{if}(b, x, x') = \begin{cases} x & \text{if } b = \mathrm{true} \\ x' & \text{if } b = \mathrm{false} \end{cases}$$

Semantics of commands

$$\mathcal{C}[\![\texttt{skip}]\!] = \lambda s \in \mathrm{State}.\ s$$

This is compositionality!

$$\mathcal{C}[\![\texttt{if } B \texttt{ then } C \texttt{ else } C']\!] = \lambda s \in \mathrm{State}.\ \mathrm{if}\left(\mathcal{C}[\![B]\!](s), \mathcal{C}[\![C]\!](s), \mathcal{C}[\![C']\!](s)\right)$$

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Semantics of commands

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$$\text{where } \mathrm{if}(b, x, x') = \begin{cases} x & \text{if } b = \mathrm{true} \\ x' & \text{if } b = \mathrm{false} \end{cases}$$

$$\mathcal{C}[\![L := A]\!] = \lambda s \in \text{State}.\ s[L \mapsto \mathcal{A}[\![A]\!]\,(s)]$$

$$\text{where } s[L \mapsto n](L') = \begin{cases} n & \text{if } L' = L \\ s(L) & \text{otherwise} \end{cases}$$

Semantics of commands

$$
\begin{aligned}
\mathcal{C}[\![\texttt{skip}]\!] &= \lambda s \in \text{State}.\, s \\
\mathcal{C}[\![\texttt{if}\ B\ \texttt{then}\ C\ \texttt{else}\ C']\!] &= \lambda s \in \text{State}.\, \text{if}\,(\mathcal{C}[\![B]\!]\,(s), \mathcal{C}[\![C]\!]\,(s), \mathcal{C}[\![C']\!]\,(s)) \\
&\quad \text{where } \text{if}(b, x, x') = \begin{cases} x & \text{if } b = \text{true} \\ x' & \text{if } b = \text{false} \end{cases} \\
\mathcal{C}[\![L := A]\!] &= \lambda s \in \text{State}.\, s[L \mapsto \mathcal{A}[\![A]\!]\,(s)] \\
&\quad \text{where } s[L \mapsto n](L') = \begin{cases} n & \text{if } L' = L \\ s(L) & \text{otherwise} \end{cases} \\
\mathcal{C}[\![C; C']\!] &= \mathcal{C}[\![C']\!] \circ \mathcal{C}[\![C]\!] \\
&= \lambda s \in \text{State}.\, \mathcal{C}[\![C']\!]\,(\mathcal{C}[\![C]\!]\,(s))
\end{aligned}
$$

Introduction

A semantics for loops

Semantics of loops?

This is all very nice, but...

$$[\![\texttt{while}\ B\ \texttt{do}\ C]\!] = ???$$

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$$[\![\texttt{while}\ B\ \texttt{do}\ C]\!] = ???$$

Remember:

- $(\texttt{while}\ B\ \texttt{do}\ C, s) \to (\texttt{if}\ B\ \texttt{then}\ (C; \texttt{while}\ B\ \texttt{do}\ C)\ \texttt{else}\ \texttt{skip}, s)$
- we want a *compositional* semantic: we should give $[\![\texttt{while}\ B\ \texttt{do}\ C]\!]$ in terms of $[\![C]\!]$ and $[\![B]\!]$

$$\begin{aligned}
[\![\texttt{while}\ B\ \texttt{do}\ C]\!] &= [\![\texttt{if}\ B\ \texttt{then}\ (C; \texttt{while}\ B\ \texttt{do}\ C)\ \texttt{else}\ \texttt{skip}]\!] \\
&= \lambda s \in \mathrm{State}.\ \mathrm{if}([\![B]\!], [\![\texttt{while}\ B\ \texttt{do}\ C]\!] \circ [\![C]\!](s), s)
\end{aligned}$$

Loop as a fixpoint

$$\begin{aligned}
[\![\texttt{while } B \texttt{ do } C]\!] &= [\![\texttt{if } B \texttt{ then } (C; \texttt{while } B \texttt{ do } C) \texttt{ else skip}]\!] \\
&= \lambda s \in \mathrm{State}.\ \mathrm{if}([\![B]\!], [\![\texttt{while } B \texttt{ do } C]\!] \circ [\![C]\!](s), s)
\end{aligned}$$

Not a direct definition for $[\![\texttt{while } B \texttt{ do } C]\!]$... But a fixed point equation!

$$[\![\texttt{while } B \texttt{ do } C]\!] = F_{[\![B]\!],[\![C]\!]}(\texttt{while } B \texttt{ do } C)$$

where $\quad F_{b,c} : \begin{array}{rcl} (\mathrm{State} \rightharpoonup \mathrm{State}) & \to & (\mathrm{State} \rightharpoonup \mathrm{State}) \\ w & \mapsto & \lambda s \in \mathrm{State}.\ \mathrm{if}(b(s), w \circ c(s), s). \end{array}$

Now we have a goal

- Why/when does $w = F_{b,c}(w)$ have a solution?
- What if it has several solutions? Which one should be our $[\![\texttt{while } B \texttt{ do } C]\!]$?

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Our occupation for the next few lectures...

Introduction

A taste of domain theory

An example

$$[\![\texttt{while}\ X > 0\ \texttt{do}\ (Y := X * Y; X := X - 1)]\!]$$

$$[\![\texttt{while } X > 0 \texttt{ do } (Y := X * Y; X := X - 1)]\!]$$

should be some w such that:

$$w = F_{[\![X>0]\!],[\![Y:=X*Y;X:=X-1]\!]}(w).$$

$$[\![\texttt{while } X > 0 \texttt{ do } (Y := X * Y; X := X - 1)]\!]$$

should be some w such that:

$$w = F_{[\![X>0]\!],[\![Y:=X*Y;X:=X-1]\!]}(w).$$

That is, we are looking for a fixed point of the following $F : D \to D$, where D is $(\text{State} \rightharpoonup \text{State})$:

$$F(w)([X \mapsto x, Y \mapsto y]) = \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ w([X \mapsto x - 1, Y \mapsto x \cdot y]) & \text{if } x > 0. \end{cases}$$

The poset of partial functions

Partial order $\sqsubseteq$ on D ($= \mathrm{State} \rightharpoonup \mathrm{State}$):

$w \sqsubseteq w'$ if for all $s \in \mathrm{State}$, if w is defined at s then so is w' and moreover $w(s) = w'(s)$.

if the graph of w is included in the graph of w'.

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if the graph of w is included in the graph of w'.

Least element $\perp \in D$:

$\perp$ = totally undefined partial function

= partial function with empty graph

Approximating the fixed point

Define $w_n = F^n(w)$, that is $\begin{cases} w_0 & = \bot \\ w_{n+1} & = F(w_n) \end{cases}.$

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$$w_1[X \mapsto x, Y \mapsto y] = F(\bot)[X \mapsto x, Y \mapsto y] = \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ \text{undefined} & \text{if } x \geq 1 \end{cases}$$

Define $w_n = F^n(w)$, that is $\begin{cases} w_0 & = \bot \\ w_{n+1} & = F(w_n) \end{cases}$.

$$w_2[X \mapsto x, Y \mapsto y] = F(w_1)[X \mapsto x, Y \mapsto y] = \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ [X \mapsto 0, Y \mapsto y] & \text{if } x = 1 \\ \text{undefined} & \text{if } x \geq 2 \end{cases}$$

Approximating the fixed point

Define $w_n = F^n(w)$, that is $\begin{cases} w_0 & = \bot \\ w_{n+1} & = F(w_n) \end{cases}$.

$$w_3[X \mapsto x, Y \mapsto y] = F(w_2)[X \mapsto x, Y \mapsto y] = \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ [X \mapsto 0, Y \mapsto y] & \text{if } x = 1 \\ [X \mapsto 0, Y \mapsto 2y] & \text{if } x = 2 \\ \text{undefined} & \text{if } x \geq 3 \end{cases}$$

Define $w_n = F^n(w)$, that is $\begin{cases} w_0 & = \bot \\ w_{n+1} & = F(w_n) \end{cases}$.

$$w_n[X \mapsto x, Y \mapsto y] = \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x < 0 \\ [X \mapsto 0, Y \mapsto (x!) \cdot y] & \text{if } 0 \leq x < n \\ \text{undefined} & \text{if } x \geq n \end{cases}$$

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$$w_0 \sqsubseteq w_1 \sqsubseteq \ldots \sqsubseteq w_n \sqsubseteq \ldots$$

Approximating the fixed point

Define $w_n = F^n(w)$, that is $\begin{cases} w_0 & = \bot \\ w_{n+1} & = F(w_n) \end{cases}$.

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$$w_0 \sqsubseteq w_1 \sqsubseteq \ldots \sqsubseteq w_n \sqsubseteq \ldots \sqsubseteq w_\infty?$$

Approximating the fixed point

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$$w_n[X \mapsto x, Y \mapsto y] = \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x < 0 \\ [X \mapsto 0, Y \mapsto (x!) \cdot y] & \text{if } 0 \leq x < n \\ \text{undefined} & \text{if } x \geq n \end{cases}$$

$$w_0 \sqsubseteq w_1 \sqsubseteq \ldots \sqsubseteq w_n \sqsubseteq \ldots \sqsubseteq w_\infty$$

$$w_\infty[X \mapsto x, Y \mapsto y] = \bigsqcup_{i \in \mathbb{N}} w_i = \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x < 0 \\ [X \mapsto 0, Y \mapsto (x!) \cdot y] & \text{if } x \geq 0 \end{cases}$$

We have our semantics

$$F(w_\infty)[X \mapsto x, Y \mapsto y]$$

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$$= \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ [X \mapsto 0, Y \mapsto (x-1)! \cdot x \cdot y] & \text{if } x > 0 \end{cases} \qquad \text{(by definition of } w_\infty\text{)}$$

We have our semantics

$$
\begin{aligned}
F(w_\infty)[X \mapsto x, Y \mapsto y] &= \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ w_\infty[X \mapsto x-1, Y \mapsto x \cdot y] & \text{if } x > 0 \end{cases} && \text{(by definition of } F\text{)} \\
&= \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ [X \mapsto 0, Y \mapsto (x-1)! \cdot x \cdot y] & \text{if } x > 0 \end{cases} && \text{(by definition of } w_\infty\text{)} \\
&= w_\infty[X \mapsto x, Y \mapsto y]
\end{aligned}
$$

We have our semantics

$$F(w_\infty)[X \mapsto x, Y \mapsto y] = \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ w_\infty[X \mapsto x-1, Y \mapsto x \cdot y] & \text{if } x > 0 \end{cases} \qquad \text{(by definition of } F\text{)}$$

$$= \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ [X \mapsto 0, Y \mapsto (x-1)! \cdot x \cdot y] & \text{if } x > 0 \end{cases} \qquad \text{(by definition of } w_\infty\text{)}$$

$$= w_\infty[X \mapsto x, Y \mapsto y]$$

- w_∞ is a fixed point
- which moreover agrees with the operational semantics (!)

Least Fixed Points

Least Fixed Points

Posets and monotone functions

Partially ordered set

A partial order on a set D is a binary relation $\sqsubseteq$ that is

reflexive: $\forall d \in D.\ d \sqsubseteq d$

transitive: $\forall d, d', d'' \in D.\ d \sqsubseteq d' \sqsubseteq d'' \Rightarrow d \sqsubseteq d''$

antisymmetric: $\forall d, d' \in D.\ d \sqsubseteq d' \sqsubseteq d \Rightarrow d = d'$.

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antisymmetric: $\forall d, d' \in D.\ d \sqsubseteq d' \sqsubseteq d \Rightarrow d = d'$.

$$\text{REFL}\ \frac{}{x \sqsubseteq x} \qquad \text{TRANS}\ \frac{x \sqsubseteq y \qquad y \sqsubseteq z}{x \sqsubseteq z} \qquad \text{ASYM}\ \frac{x \sqsubseteq y \qquad y \sqsubseteq x}{x = y}$$

Domain of partial functions $X \rightharpoonup Y$

Underlying set: partial functions f with domain of definition $\mathbf{dom}(f) \subseteq X$ and taking values in Y;

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Underlying set: partial functions f with domain of definition $\mathrm{dom}(f) \subseteq X$ and taking values in Y;

Order: $f \sqsubseteq g$ if $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$ and $\forall x \in \mathrm{dom}(f).\ f(x) = g(x)$, *i.e.* if $\mathrm{graph}(f) \subseteq \mathrm{graph}(g)$.

A function $f\colon D \to E$ between posets is **monotone** if

$$\forall d, d' \in D.\, d \sqsubseteq d' \Rightarrow f(d) \sqsubseteq f(d').$$

A function $f\colon D \to E$ between posets is monotone if

$$\forall d, d' \in D.\, d \sqsubseteq d' \Rightarrow f(d) \sqsubseteq f(d').$$

$$\textsc{Mon}\ \frac{x \sqsubseteq y}{f(x) \sqsubseteq f(y)}$$

Least Fixed Points

Least elements and pre-fixed points

Least element

An element $d \in S$ is the least element of S if it satisfies

$$\forall x \in S.\, d \sqsubseteq x.$$

Least element

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$$\forall x \in S.\ d \sqsubseteq x.$$

If it exists, it is unique , and is written $\bot_S$, or simply $\bot$.

$$\textsc{Least}\ \frac{x \in S}{\bot_S \sqsubseteq x}$$

An element $d \in S$ is the **least** element of S if it satisfies

$$\forall x \in S.\ d \sqsubseteq x.$$

If it exists, it is unique , and is written $\bot_S$, or simply $\bot$.

$$\textsc{Least}\ \frac{x \in S}{\bot_S \sqsubseteq x} \qquad \textsc{ASym}\ \frac{\textsc{Least}\ \dfrac{\bot'_S \in S}{\bot_S \sqsubseteq \bot'_S} \qquad \textsc{Least}\ \dfrac{\bot_S \in S}{\bot'_S \sqsubseteq \bot_S}}{\bot_S = \bot'_S}$$

Pre-fixed point

An element $d \in D$ is a pre-fixed point of f if it satisfies $f(d) \sqsubseteq d$.

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It is thus (uniquely) specified by the two properties:

$$\text{LFP-FIX}\ \frac{}{f(\mathrm{fix}(f)) \sqsubseteq \mathrm{fix}(f)} \qquad \text{LFP-LEAST}\ \frac{f(d) \sqsubseteq d}{\mathrm{fix}(f) \sqsubseteq d}$$

$$\textsc{lfp-fix}\ \frac{}{f(\mathrm{fix}(f)) \sqsubseteq \mathrm{fix}(f)}$$

The least pre-fixed point is a fixed point.

$$\textsc{lfp-fix}\ \frac{}{f(\mathrm{fix}(f)) \sqsubseteq \mathrm{fix}(f)} \qquad \textsc{lfp-least}\ \frac{f(d) \sqsubseteq d}{\mathrm{fix}(f) \sqsubseteq d}$$

To prove $\mathrm{fix}(f) \sqsubseteq d$, it is enough to show $f(d) \sqsubseteq d$.

$$\text{LFP-FIX}\ \frac{}{f(\mathrm{fix}(f)) \sqsubseteq \mathrm{fix}(f)} \qquad \text{LFP-LEAST}\ \frac{f(d) \sqsubseteq d}{\mathrm{fix}(f) \sqsubseteq d}$$

Application: least pre-fixed points of monotone functions are (least) fixed points.

$$\text{ASYM}\ \frac{\text{LFP-FIX}\ \dfrac{}{f(\mathrm{fix}(f)) \sqsubseteq \mathrm{fix}(f)} \qquad \dfrac{}{\mathrm{fix}(f) \sqsubseteq f(\mathrm{fix}(f))}}{f(\mathrm{fix}(f)) = \mathrm{fix}(f)}$$

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Application: least pre-fixed points of monotone functions are (least) fixed points.

$$\textsc{ASym}\ \frac{\textsc{lfp-fix}\ \dfrac{}{f(\mathrm{fix}(f)) \sqsubseteq \mathrm{fix}(f)} \qquad \textsc{lfp-least}\ \dfrac{\textsc{Mon}\ \dfrac{\textsc{lfp-fix}\ \dfrac{}{f(\mathrm{fix}(f)) \sqsubseteq \mathrm{fix}(f)}}{f(f(\mathrm{fix}(f))) \sqsubseteq f(\mathrm{fix}(f))}}{\mathrm{fix}(f) \sqsubseteq f(\mathrm{fix}(f))}}{f(\mathrm{fix}(f)) = \mathrm{fix}(f)}$$

Least Fixed Points

Least upper bounds

The **least upper bound** of countable increasing chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$, written $\bigsqcup_{n\geq 0} d_n$, satisfies the two following properties:

$$\text{LUB-BOUND}\ \frac{}{x_i \sqsubseteq \bigsqcup_{n\geq 0} x_n} \qquad \text{LUB-LEAST}\ \frac{\forall n \geq 0 \,.\, x_n \sqsubseteq x}{\bigsqcup_{n\geq 0} x_n \sqsubseteq x}$$

Properties of lubs

Lubs are unique.

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Lubs are monotone: if for all $n \in \mathbb{N}.\ d_n \sqsubseteq e_n$, then $\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n$.

Properties of lubs

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Lubs are monotone: if for all $n \in \mathbb{N}.\ d_n \sqsubseteq e_n$, then $\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n$.

$$\textsc{lub-mon}\ \frac{\forall i.\ d_i \sqsubseteq e_i}{\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n}$$

Properties of lubs

Lubs are unique.

Lubs are monotone: if for all $n \in \mathbb{N}.\ d_n \sqsubseteq e_n$, then $\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n$.

For any d, $\bigsqcup_n d = d$.

Lubs are unique.

Lubs are monotone: if for all $n \in \mathbb{N}.\ d_n \sqsubseteq e_n$, then $\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n$.

For any d, $\bigsqcup_n d = d$.

For any chain and $N \in \mathbb{N}$, $\bigsqcup_n d_n = \bigsqcup_n d_{n+N}$.

Properties of lubs

Lubs are unique (if they exist).

Lubs are monotone: if for all $n \in \mathbb{N}.\ d_n \sqsubseteq e_n$, then $\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n$ (if they exist).

For any d, $\bigsqcup_n d = d$ (and in particular it exists).

For any chain and $N \in \mathbb{N}$, $\bigsqcup_n d_n = \bigsqcup_n d_{n+N}$ (if any of the two exists).

Diagonalisation

Assume $d_{m,n} \in D$ $(m, n \geq 0)$ satisfies

$$m \leq m' \wedge n \leq n' \Rightarrow d_{m,n} \sqsubseteq d_{m',n'}.$$

Assume $d_{m,n} \in D$ $(m, n \geq 0)$ satisfies

$$m \leq m' \wedge n \leq n' \Rightarrow d_{m,n} \sqsubseteq d_{m',n'}. \qquad (\dagger)$$

Then, assuming they exist, the lubs form two chains

$$\bigsqcup_{n \geq 0} d_{0,n} \sqsubseteq \bigsqcup_{n \geq 0} d_{1,n} \sqsubseteq \bigsqcup_{n \geq 0} d_{2,n} \sqsubseteq \ldots$$

and

$$\bigsqcup_{m \geq 0} d_{m,0} \sqsubseteq \bigsqcup_{m \geq 0} d_{m,1} \sqsubseteq \bigsqcup_{m \geq 0} d_{m,2} \sqsubseteq \ldots$$

Assume $d_{m,n} \in D$ $(m, n \geq 0)$ satisfies

$$m \leq m' \land n \leq n' \Rightarrow d_{m,n} \sqsubseteq d_{m',n'}. \tag{†}$$

Then, assuming they exist, the lubs form two chains

$$\bigsqcup_{n\geq 0} d_{0,n} \sqsubseteq \bigsqcup_{n\geq 0} d_{1,n} \sqsubseteq \bigsqcup_{n\geq 0} d_{2,n} \sqsubseteq \ldots$$

and

$$\bigsqcup_{m\geq 0} d_{m,0} \sqsubseteq \bigsqcup_{m\geq 0} d_{m,1} \sqsubseteq \bigsqcup_{m\geq 0} d_{m,2} \sqsubseteq \ldots$$

Moreover, again assuming they exist,

$$\bigsqcup_{m\geq 0} \left(\bigsqcup_{n\geq 0} d_{m,n} \right) = \bigsqcup_{k\geq 0} d_{k,k} = \bigsqcup_{n\geq 0} \left(\bigsqcup_{m\geq 0} d_{m,n} \right) .$$

Least Fixed Points

Complete partial orders and domains

A chain complete poset/cpo is a poset $(D, \sqsubseteq)$ in which all chains have least upper bounds.

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Beware: the lub need only exist if the x_i form a chain!

Cpos and domains

A chain complete poset/cpo is a poset $(D, \sqsubseteq)$ in which all chains have least upper bounds.

Beware: the lub need only exist if the x_i form a chain!

A domain is a cpo with a least element $\bot$.

Least element: $\bot$ is the totally undefined function.

Domain of partial functions

Least element: $\bot$ is the totally undefined function.

Lub of a chain: $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \ldots$ has lub f such that

$$f(x) = \begin{cases} f_n(x) & \text{if } x \in \mathbf{dom}(f_n) \text{ for some } n \\ \text{undefined} & \text{otherwise} \end{cases}$$

Least element: $\bot$ is the totally undefined function.

Lub of a chain: $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \ldots$ has lub f such that

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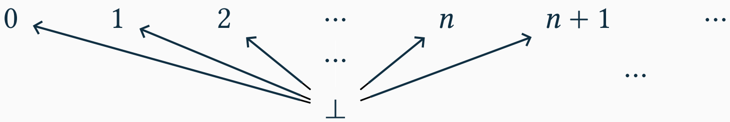
Beware: the definition of $\bigsqcup_{n \geq 0} f_n$ is unambiguous only if the f_i form a chain!


0
1
2
...
n
n + 1
...
...
...
⊥

Least Fixed Points

Continuous functions

Given two cpos D and E, a function $f\colon D \to E$ is continuous if

- it is monotone, and
- it preserves lubs of chains, *i.e.* for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \dots$ in D, we have

$$f(\bigsqcup_{n\geq 0} d_n) = \bigsqcup_{n\geq 0} f(d_n)$$

Given two cpos D and E, a function $f : D \to E$ is continuous if

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$$f(\bigsqcup_{n \geq 0} d_n) = \bigsqcup_{n \geq 0} f(d_n)$$

A function f is strict if $f(\bot_D) = \bot_E$.

All computable functions are continuous.

Thesis

All computable functions are continuous.

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The typical non-continuous function: “is a sequence the constant $\mathbf{0}$”?

$$
\begin{array}{llllll}
0 & 0 & \bot & \dots & & \mapsto \bot \\
0 & 0 & 0 & 0 & 1 \quad \dots & \mapsto 1
\end{array}
$$

$$
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \overline{0} \qquad \mapsto 0
$$

All computable functions are continuous.

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$$
\begin{array}{llllll}
0 & 0 & \bot & \ldots & & \mapsto \bot \\
0 & 0 & 0 & 0 & 1 \;\ldots & \mapsto 1 \\
0 & 0 & 0 & 0 & 0 \;\ldots & \mapsto ? \\
\\
0 & 0 & 0 & 0 & 0 \;\overline{0} & \mapsto 0
\end{array}
$$

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$$
\begin{array}{llllllllllr}
0 & 0 & \bot & \dots & & & & & & & \mapsto \bot \\
0 & 0 & 0 & 0 & 1 & \dots & & & & & \mapsto 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bot & \dots & \mapsto \bot \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & \mapsto ? \\
0 & 0 & 0 & 0 & 0 & \overline{0} & & & & & \mapsto 0
\end{array}
$$

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$$
\begin{array}{llllllllllr}
0 & 0 & \bot & \dots & & & & & & & \mapsto \bot \\
0 & 0 & 0 & 0 & 1 & \dots & & & & & \mapsto 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bot & \dots & \mapsto \bot \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & \mapsto ? \\
0 & 0 & 0 & 0 & 0 & \overline{0} & & & & & \mapsto 0
\end{array}
$$

Intuition: non-continuity $\approx$ “jump at infinity” $\approx$ non-computability

All computable functions are continuous.

The typical non-continuous function: “is a sequence the constant $\mathbf{0}$”?

$$
\begin{array}{cccccccccccl}
0 & 0 & \bot & \dots & & & & & & & \mapsto & \bot \\
0 & 0 & 0 & 0 & 1 & \dots & & & & & \mapsto & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bot & \dots & \mapsto & \bot \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & \mapsto & ? \\
0 & 0 & 0 & 0 & 0 & \overline{0} & & & & & \mapsto & 0
\end{array}
$$

Intuition: non-continuity ≈ “jump at infinity” ≈ non-computability

Later in the course: show the thesis… by giving a denotational semantics.

Least Fixed Points

Kleene's fixed point theorem

Let $f\colon D \to D$ be a continuous function on a domain D. Then f possesses a least pre-fixed point, given by

$$\mathrm{fix}(f) = \bigsqcup_{n \geq 0} f^n(\bot).$$

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Let $f\colon D \to D$ be a continuous function on a domain D. Then f possesses a least pre-fixed point, given by

$$\mathrm{fix}(f) = \bigsqcup_{n \geq 0} f^n(\bot).$$

It is thus also the least fixed point of f!

Constructions on Domains

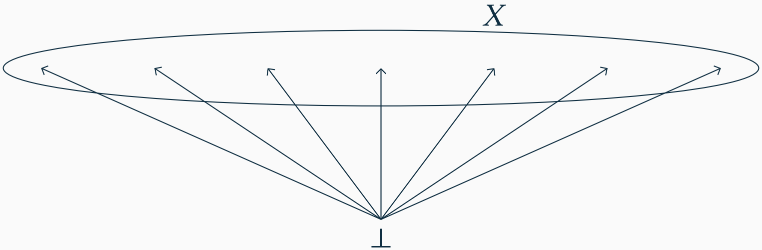
Constructions on Domains

Flat domains

Flat domain on X

The flat domain on a set X is defined by:

- its underlying set $X \biguplus \{\bot\}$;
- $x \sqsubseteq x'$ if either $x = \bot$ or $x = x'$.



Let $f : X \rightharpoonup Y$ be a partial function between two sets. Then

$$\begin{aligned} f_\perp : \quad X_\perp \quad &\to \quad Y_\perp \\ d \quad &\mapsto \begin{cases} f(d) & \text{if } d \in X \text{ and } f \text{ is defined at } d \\ \perp & \text{if } d \in X \text{ and } f \text{ is not defined at } d \\ \perp & \text{if } d = \perp \end{cases} \end{aligned}$$

defines a continuous function between the corresponding flat domains.

Constructions on Domains

Products of domains

The product of two posets $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ has underlying set

$$D_1 \times D_2 = \{(d_1, d_2) \mid d_1 \in D_1 \wedge d_2 \in D_2\}$$

and partial order $\sqsubseteq$ defined by

$$(d_1, d_2) \sqsubseteq (d_1', d_2') \overset{\text{def}}{\Leftrightarrow} d_1 \sqsubseteq_1 d_1' \wedge d_2 \sqsubseteq_2 d_2'$$

Binary product

The product of two posets $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ has underlying set

$$D_1 \times D_2 = \{(d_1, d_2) \mid d_1 \in D_1 \wedge d_2 \in D_2\}$$

and partial order $\sqsubseteq$ defined by

$$(d_1, d_2) \sqsubseteq (d_1', d_2') \overset{\text{def}}{\Leftrightarrow} d_1 \sqsubseteq_1 d_1' \wedge d_2 \sqsubseteq_2 d_2'$$

$$\text{POX}\ \frac{d_1 \sqsubseteq_1 d_1' \qquad d_2 \sqsubseteq_2 d_2'}{(d_1, d_2) \sqsubseteq (d_1', d_2')}$$

Componentwise lubs and least elements

lubs of chains are computed componentwise:

$$\bigsqcup_{n \geq 0} (d_{1,n}, d_{2,n}) = (\bigsqcup_{i \geq 0} d_{1,i}, \bigsqcup_{j \geq 0} d_{2,j}).$$

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$$\bigsqcup_{n \geq 0} (d_{1,n}, d_{2,n}) = (\bigsqcup_{i \geq 0} d_{1,i}, \bigsqcup_{j \geq 0} d_{2,j}).$$

If $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ have least elements, so does $(D_1 \times D_2, \sqsubseteq)$ with

$$\bot_{D_1 \times D_2} = (\bot_{D_1}, \bot_{D_2})$$

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$$\bot_{D_1\times D_2} = (\bot_{D_1}, \bot_{D_2})$$

Products of cpos (domains) are cpos (domains).

Functions of two arguments

A function $f : (D \times E) \to F$ is monotone if and only if it is monotone in each argument separately:

$$\forall d, d' \in D, e \in E.\, d \sqsubseteq d' \Rightarrow f(d, e) \sqsubseteq f(d', e)$$
$$\forall d \in D, e, e' \in E.\, e \sqsubseteq e' \Rightarrow f(d, e) \sqsubseteq f(d, e').$$

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$$\forall d \in D, e, e' \in E.\, e \sqsubseteq e' \Rightarrow f(d,e) \sqsubseteq f(d,e').$$

Moreover, it is continuous if and only if it preserves lubs in each argument separately:

$$f(\bigsqcup_{m \geq 0} d_m\, ,\, e) = \bigsqcup_{m \geq 0} f(d_m, e)$$
$$f(d\, ,\, \bigsqcup_{n \geq 0} e_n) = \bigsqcup_{n \geq 0} f(d, e_n).$$

Derived rules for functions of two arguments

$$\text{MON}\times\ \frac{f \text{ monotone} \qquad x \sqsubseteq x' \qquad y \sqsubseteq y'}{f(x,y) \sqsubseteq f(x',y')}$$

$$f\left(\bigsqcup_m x_m, \bigsqcup_n y_n\right) = \bigsqcup_m \bigsqcup_n f(x_m, y_n) = \bigsqcup_k f(x_k, y_k)$$

Let D_1 and D_2 be cpos. The projections

$$\begin{aligned} \pi_1 : \quad D_1 \times D_2 &\to D_1 \\ (d_1, d_2) &\mapsto d_1 \end{aligned} \qquad \begin{aligned} \pi_2 : \quad D_1 \times D_2 &\to D_2 \\ (d_1, d_2) &\mapsto d_2 \end{aligned}$$

are continuous functions.

Let D_1 and D_2 be cpos. The **projections**

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are continuous functions.

If $f_1 : D \to D_1$ and $f_2 : D \to D_2$ are continuous functions from a cpo D, then the **pairing** function

$$\begin{array}{rrcl} \langle f_1, f_2 \rangle : & D & \to & D_1 \times D_2 \\ & d & \mapsto & (f_1(d), f_2(d)) \end{array}$$

is continuous.

The **conditional** function

$$\begin{aligned} \mathrm{if} : \quad \mathbb{B}_\perp \times (D \times D) &\to D \\ (x, d) &\mapsto \begin{cases} \pi_1(d) & \text{if } x = \mathrm{true} \\ \pi_2(d) & \text{if } x = \mathrm{false} \\ \perp_D & \text{if } x = \perp \end{cases} \end{aligned}$$

is continuous.

Given a set I, suppose that for each $i \in I$ we are given a set X_i. The (cartesian) **product** of the X_i is

$$\prod_{i \in I} X_i$$

Two ways to see it:

- tuples: $(\dots, x_i, \dots)_{i \in I}$ such that $x_i \in X_i$;

General product

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Special case: $\prod_{i \in \mathbb{B}} D_i$ corresponds to $D_{\mathrm{true}} \times D_{\mathrm{false}}$.

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Special case: $\prod_{i \in \mathbb{B}} D_i$ corresponds to $D_{\mathrm{true}} \times D_{\mathrm{false}}$.

Projections (for any $i \in I$):

$$\pi_i : \left(\prod_{i \in I} X_i \right) \to X_i$$

Given a set I, suppose that for each $i \in I$ we are given a cpo $(D_i, \sqsubseteq_i)$. The **product** of this whole family of cpos has

- underlying set equal to $\prod_{i \in I} D_i$;

General product of domains

Given a set I, suppose that for each $i \in I$ we are given a cpo $(D_i, \sqsubseteq_i)$. The product of this whole family of cpos has

- underlying set equal to $\prod_{i \in I} D_i$;
- componentwise order

$$p \sqsubseteq p' \overset{\text{def}}{\Leftrightarrow} \forall i \in I \,.\, p_i \sqsubseteq_i p'_i \,.$$

Given a set I, suppose that for each $i \in I$ we are given a cpo $(D_i, \sqsubseteq_i)$. The **product** of this whole family of cpos has

- underlying set equal to $\prod_{i \in I} D_i$;
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$$p \sqsubseteq p' \overset{\text{def}}{\Leftrightarrow} \forall i \in I \,.\, p_i \sqsubseteq_i p'_i \,.$$

I-indexed products of cpos (domains) are cpos (domains), and projections are continuous.

Constructions on Domains

Function domains

Given two cpos $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$, the **function cpo** $(D \to E, \sqsubseteq)$ has underlying set

$$\{f : D \to E \mid \text{ is a } \textit{continuous} \text{ function}\}$$

equipped with the pointwise order:

$$f \sqsubseteq f' \overset{\text{def}}{\Leftrightarrow} \forall d \in D.\ f(d) \sqsubseteq_E f'(d).$$

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$$f \sqsubseteq f' \overset{\text{def}}{\Leftrightarrow} \forall d \in D.\, f(d) \sqsubseteq_E f'(d).$$

$$\frac{f \sqsubseteq_{D \to E} g \qquad x \sqsubseteq_D y}{f(x) \sqsubseteq_E g(y)}$$

Given two cpos $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$, the function cpo $(D \to E, \sqsubseteq)$ has underlying set

$$\{f : D \to E \mid \text{ is a } \textit{continuous} \text{ function}\}$$

equipped with the pointwise order:

$$f \sqsubseteq f' \overset{\text{def}}{\Leftrightarrow} \forall d \in D.\ f(d) \sqsubseteq_E f'(d).$$

Argumentwise least elements and lubs:

$$\bot_{D \to E}(d) = \bot_E \qquad \left(\bigsqcup_{n \geq 0} f_n\right)(d) = \bigsqcup_{n \geq 0} f_n(d)$$

Evaluation, currying ($f : (D' \times D) \to E$) and composition

$$\begin{array}{rccc} \mathrm{eval} : & (D \to E) \times D & \to & E \\ & (f, d) & \mapsto & f(d) \end{array}$$

$$\begin{array}{rccc} \mathrm{cur}(f) : & D' & \to & (D \to E) \\ & d' & \mapsto & \lambda d \in D.\ f(d', d) \end{array}$$

$$\begin{array}{rccc} \circ : & \big((E \to F) \times (D \to E)\big) & \longrightarrow & (D \to F) \\ & (f, g) & \mapsto & \lambda d \in D.\ g(f(d)) \end{array}$$

are all well-defined and continuous.

$$\text{fix}\colon \quad (D \to D) \quad \to \quad D$$

is continuous.

Constructions on Domains

Back to the introduction

$$[\![\texttt{while}\ X > 0\ \texttt{do}\ (Y := X * Y; X := X - 1)]\!]$$

is a fixed point of the following $F : D \to D$, where D is $(\mathbf{State} \rightharpoonup \mathbf{State})$:

$$F(w)([X \mapsto x, Y \mapsto y]) \quad = \quad \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ w([X \mapsto x - 1, Y \mapsto x \cdot y]) & \text{if } x > 0. \end{cases}$$

The semantics of a while loop

$$[\![\texttt{while } X > 0 \texttt{ do } (Y := X * Y; X := X - 1)]\!]$$

is a fixed point of the following $F : D \to D$, where D is $(\text{State}_\bot \to \text{State}_\bot)$:

$$F(w)([X \mapsto x, Y \mapsto y]) = \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ w([X \mapsto x - 1, Y \mapsto x \cdot y]) & \text{if } x > 0. \end{cases}$$

$$F(\bot) = \bot$$

$\text{State}_\bot \to \text{State}_\bot$ is a domain!

Kleene's fixed point theorem:

$$w_\infty = \bigsqcup_{i \in \mathbb{N}} F^n(\bot)$$

is the least fixed point of F, and in particular a fixed point.

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is the least fixed point of F, and in particular a fixed point.

We **can** compute explicitly

$$w_\infty[X \mapsto x, Y \mapsto y] = \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x < 0 \\ [X \mapsto 0, Y \mapsto (x!) \cdot y] & \text{if } x \geq 0 \end{cases}$$

And **check** this agrees with the operational semantics.

Scott Induction

Let D be a domain, $f\colon D \to D$ be a continuous function and $S \subseteq D$ be a subset of D. If the set S

(i) contains $\bot$,

(ii) is stable under f, *i.e.* $f(S) \subseteq S$,

(iii) is chain-closed, *i.e.* the lub of any chain of elements of S is also in S,

then $\mathrm{fix}(f) \in S$.

Reasoning on fixed points: Scott induction

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$$\textsc{ScottInd}\ \frac{\Phi(\bot) \qquad \Phi(x) \Rightarrow \Phi(f(x)) \qquad (\forall i \in \mathbb{N}.\ \Phi(x_i)) \Rightarrow \Phi(\bigsqcup_{i \in \mathbb{N}} x_i)}{\Phi(\mathrm{fix}(f))}$$

All the following are chain-closed:

Building chain-closed sets

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$f^{-1}S = \{x \in D \mid f(x) \in S\}$ if $S \subseteq E$ is chain-closed, and $f : D \to E$ is continuous

$S \cup T$ and $\bigcap_{i \in I} S_i$ if S, T and S_i are

$\forall S \stackrel{\text{def}}{=} \{y \in E \mid \forall x \in D.\, (x, y) \in S\} \subseteq E$ if $S \subseteq D \times E$ is

Example: downset

Assume $f(d) \sqsubseteq d$, *i.e.* d is a pre-fixed point of the continuous $f : D \to D$. By Scott induction on $d \downarrow$, $\mathrm{fix}(f) \sqsubseteq d$.

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Proof!

Example: partial correctness

Let $w_\infty \colon \mathrm{State}_\perp \to \mathrm{State}_\perp$ be the denotation of

$$\texttt{while } X > 0 \texttt{ do } (Y := X * Y; X := X - 1)$$

Recall that $w_\infty = \mathrm{fix}(F)$ where

$$F(w)(x, y) = \begin{cases} (x, y) & \text{if } x \leq 0 \\ w(x - 1, x \cdot y) & \text{if } x > 0 \end{cases}$$

$$F(w)(\perp) = \perp$$

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Proof: by Scott induction!

PCF

PCF

Terms and types

Syntax of PCF

Types: $\tau ::= \texttt{nat} \mid \texttt{bool} \mid \tau \to \tau$

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Terms:

$$
\begin{aligned}
t \quad ::= \quad & \texttt{0} \mid \texttt{succ}(t) \mid \texttt{pred}(t) \mid \\
& \texttt{true} \mid \texttt{false} \mid \texttt{zero?}(t) \mid \texttt{if}\ t\ \texttt{then}\ t\ \texttt{else}\ t \\
& x \mid \texttt{fun}\, x{:}\,\tau.\, t \mid t\ t \mid \texttt{fix}(t)
\end{aligned}
$$

Typing for PCF (I)

$\boxed{\Gamma \vdash t : \tau}$ The term t has type τ in context Γ

$$\textsc{Zero}\ \frac{}{\Gamma \vdash \mathtt{0} : \mathtt{nat}} \qquad \textsc{Succ}\ \frac{\Gamma \vdash t : \mathtt{nat}}{\Gamma \vdash \mathtt{succ}(t) : \mathtt{nat}} \qquad \textsc{Pred}\ \frac{\Gamma \vdash t : \mathtt{nat}}{\Gamma \vdash \mathtt{pred}(t) : \mathtt{nat}}$$

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$$\textsc{True}\ \frac{}{\Gamma \vdash \mathtt{true} : \mathtt{bool}} \qquad \textsc{False}\ \frac{}{\Gamma \vdash \mathtt{false} : \mathtt{bool}} \qquad \textsc{IsZ}\ \frac{\Gamma \vdash t : \mathtt{nat}}{\Gamma \vdash \mathtt{zero?}(t) : \mathtt{bool}}$$

$$\textsc{If}\ \frac{\Gamma \vdash b : \mathtt{bool} \qquad \Gamma \vdash t : \tau \qquad \Gamma \vdash t' : \tau}{\Gamma \vdash \mathtt{if}\ b\ \mathtt{then}\ t\ \mathtt{else}\ t' : \tau}$$

Typing for PCF (II)

$$\textsc{Var}\ \frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \qquad \textsc{Fun}\ \frac{\Gamma, x{:}\sigma \vdash t : \tau}{\Gamma \vdash \mathsf{fun}\, x{:}\sigma.\, t : \sigma \to \tau} \qquad \textsc{App}\ \frac{\Gamma \vdash f : \sigma \to \tau \qquad \Gamma \vdash u : \sigma}{\Gamma \vdash f\ u : \tau}$$

$$\textsc{Fix}\ \frac{\Gamma \vdash f : \tau \to \tau}{\Gamma \vdash \mathtt{fix}(f) : \tau}$$

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$$\textsc{Fix}\ \frac{\Gamma \vdash f : \tau \to \tau}{\Gamma \vdash \mathtt{fix}(f) : \tau}$$

$$\mathrm{PCF}_{\Gamma,\tau} \overset{\text{def}}{=} \{t \mid \Gamma \vdash t : \tau\} \qquad\qquad \mathrm{PCF}_{\tau} \overset{\text{def}}{=} \mathrm{PCF}_{\cdot,\tau}$$

PCF

Operational Semantics

PCF evaluation

Values:

$$v ::= \underbrace{\mathtt{0} \mid \mathtt{succ}(v)}_{\underline{n}} \mid \mathtt{true} \mid \mathtt{false} \mid \mathtt{fun}\, x{:}\,\tau.\, t$$

PCF evaluation

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$$\text{Val}\ \frac{\vdash v : \tau}{v \Downarrow_\tau v} \qquad \text{Succ}\ \frac{t \Downarrow_{\mathtt{nat}} v}{\mathtt{succ}(t) \Downarrow_{\mathtt{nat}} \mathtt{succ}(v)} \qquad \text{Pred}\ \frac{t \Downarrow_{\mathtt{nat}} \mathtt{succ}(v)}{\mathtt{pred}(t) \Downarrow_{\mathtt{nat}} v}$$

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$$\textsc{ZeroZ}\ \frac{t \Downarrow_{\texttt{nat}} \texttt{0}}{\texttt{zero?}(t) \Downarrow_{\texttt{bool}} \texttt{true}} \qquad \ldots \qquad \textsc{IfT}\ \frac{b \Downarrow_{\texttt{bool}} \texttt{true} \qquad t_1 \Downarrow_\tau v}{\texttt{if}\ b\ \texttt{then}\ t_1\ \texttt{else}\ t_2 \Downarrow_\tau v} \qquad \ldots$$

PCF EVALUATION

Values: $$v ::= \underbrace{\mathtt{0} \mid \mathtt{succ}(v)}_{\underline{n}} \mid \mathtt{true} \mid \mathtt{false} \mid \mathtt{fun}\, x\!:\!\tau.\, t$$

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$$\textsc{Fun}\ \frac{t \Downarrow_{\sigma\to\tau} \mathtt{fun}\, x\!:\!\sigma.\, t' \qquad t'[u/x] \Downarrow_\tau v}{t\ u \Downarrow_\tau v} \qquad \textsc{Fix}\ \frac{t\ (\mathtt{fix}(t)) \Downarrow_\tau v}{\mathtt{fix}(t) \Downarrow_\tau v}$$

PCF evaluation

Values: $v ::= \underbrace{\mathtt{0} \mid \mathtt{succ}(v)}_{\underline{n}} \mid \mathtt{true} \mid \mathtt{false} \mid \mathtt{fun}\, x{:}\,\tau.\, t$

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Alternatively: small-step $t \rightsquigarrow_\tau u$, we have $t \Downarrow_\tau v$ iff $t \rightsquigarrow_\tau^\star u$.

Examples

$$\begin{aligned}\texttt{plus} \;\stackrel{\text{def}}{=}\; & \texttt{fun}\, x\colon \texttt{nat}.\ \texttt{fix}(\texttt{fun}(p\colon \texttt{nat} \to \texttt{nat})(y\colon \texttt{nat}).\\ & \quad \texttt{if zero?}(y)\ \texttt{then}\ x\ \texttt{else succ}(p\ \texttt{pred}(y)))\end{aligned}$$

$$\texttt{plus}\ \underline{3}\ \underline{1} \Downarrow_{\texttt{nat}} \underline{4}$$

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$$\Omega_\tau \stackrel{\text{def}}{=} \texttt{fix}(\texttt{fun}\, x\colon \tau.\ x)$$

$$\Omega_\tau \Uparrow_\tau \qquad \text{(diverges)}$$

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Try it out!

PCF is Turing-complete: for every partial recursive function ϕ, there is a PCF term $\underline{\phi}$ such that for all $n \in \mathbb{N}$, if $\phi(n)$ is defined then $\underline{\phi}\ \underline{n} \Downarrow_{\mathsf{nat}} \underline{\phi(n)}$.

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(Later on: $\phi = \llbracket \underline{\phi} \rrbracket$).

Evaluation in PCF is **deterministic**: if both $t \Downarrow_\tau v$ and $t \Downarrow_\tau v'$ hold, then $v = v'$.

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By (rule) induction on evaluation $\Downarrow$:

$$\{(t, \tau, v) \mid t \Downarrow_\tau v \wedge \forall v'.(t \Downarrow_\tau v' \Rightarrow v = v')\}$$

Intuition: there is always exactly one rule which applies.

PCF

Contextual equivalence

Contextual equivalence – informal

Two phrases of a programming language are **contextually equivalent** if any occurrences of the first phrase in a **complete program** can be replaced by the second phrase without affecting the **observable results** of executing the program.

Contextual equivalence – informal

Two phrases of a programming language are **contextually equivalent** if any occurrences of the first phrase in a **complete program** can be replaced by the second phrase without affecting the **observable results** of executing the program.

The intuitive notion of **program equivalence** for programmers.

$$
\begin{aligned}
\mathcal{C} \quad ::= \quad & - \mid \mathtt{succ}(\mathcal{C}) \mid \mathtt{pred}(\mathcal{C}) \mid \mathtt{zero?}(\mathcal{C}) \mid \\
& \mathtt{if}\ \mathcal{C}\ \mathtt{then}\ t\ \mathtt{else}\ t \mid \mathtt{if}\ t\ \mathtt{then}\ \mathcal{C}\ \mathtt{else}\ t \mid \mathtt{if}\ t\ \mathtt{then}\ t\ \mathtt{else}\ \mathcal{C} \mid \\
& \mathtt{fun}\ x{:}\tau.\ \mathcal{C} \mid \mathcal{C}\ t \mid t\ \mathcal{C} \mid \mathtt{fix}(\mathcal{C})
\end{aligned}
$$

Evaluation contexts

$$
\begin{aligned}
\mathcal{C} \quad ::= \quad & - \mid \mathtt{succ}(\mathcal{C}) \mid \mathtt{pred}(\mathcal{C}) \mid \mathtt{zero?}(\mathcal{C}) \mid \\
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\end{aligned}
$$

Typing extended to evaluation contexts: $\Gamma \vdash_{\Delta,\sigma} \mathcal{C} : \tau$.

$$\begin{array}{rcl} \mathcal{C} & ::= & - \mid \texttt{succ}(\mathcal{C}) \mid \texttt{pred}(\mathcal{C}) \mid \texttt{zero?}(\mathcal{C}) \mid \\ & & \texttt{if } \mathcal{C} \texttt{ then } t \texttt{ else } t \mid \texttt{if } t \texttt{ then } \mathcal{C} \texttt{ else } t \mid \texttt{if } t \texttt{ then } t \texttt{ else } \mathcal{C} \mid \\ & & \texttt{fun } x{:}\tau.\, \mathcal{C} \mid \mathcal{C}\ t \mid t\ \mathcal{C} \mid \texttt{fix}(\mathcal{C}) \end{array}$$

Typing extended to evaluation contexts: $\Gamma \vdash_{\Delta,\sigma} \mathcal{C} : \tau$.

$$\frac{}{\Gamma \vdash_{\Gamma,\tau} - : \tau} \qquad \frac{\Gamma \vdash_{\Delta,\sigma} \mathcal{C} : \tau_1 \to \tau_2 \qquad \Gamma \vdash u : \tau_1}{\Gamma \vdash_{\Delta,\sigma} \mathcal{C}\ u : \tau_2} \qquad \ldots$$

Given a type τ, a typing context Γ and terms $t, t' \in \mathrm{PCF}_{\Gamma,\tau}$, contextual equivalence, written $\Gamma \vdash t \cong_{\mathrm{ctx}} t' : \tau$ is defined to hold if for all evaluation contexts $\mathcal{C}$ such that $\cdot \vdash_{\Gamma,\tau} \mathcal{C} : \gamma$, where γ is `nat` or `bool`, and for all values $v \in \mathrm{PCF}_\gamma$,

$$\mathcal{C}[t] \Downarrow_\gamma v \Leftrightarrow \mathcal{C}[t'] \Downarrow_\gamma v.$$

When Γ is the empty context, we simply write $t \cong_{\mathrm{ctx}} t' : \tau$ for $\cdot \vdash t \cong_{\mathrm{ctx}} t' : \tau$.

PCF

Introducing denotational semantics

The aims of denotational semantics

- a mapping of PCF types τ to domains $[\![\tau]\!]$;
- a mapping of closed, well-typed PCF terms $\cdot \vdash t : \tau$ to elements $[\![t]\!] \in [\![\tau]\!]$;
- denotation of open terms will be continuous functions.

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- denotation of open terms will be continuous functions.

Compositionality: $[\![t]\!] = [\![t']\!] \Rightarrow [\![\mathcal{C}[t]]\!] = [\![\mathcal{C}[t']]\!]$.

Soundness: for any type τ, $t \Downarrow_\tau v \Rightarrow [\![t]\!] = [\![v]\!]$.

Adequacy: for $\gamma = \texttt{bool}$ or $\texttt{nat}$, if $t \in \mathrm{PCF}_\gamma$ and $[\![t]\!] = [\![v]\!]$ then $t \Downarrow_\gamma v$.

Proof principle: to show

$$t_1 \cong_{\mathrm{ctx}} t_2 : \tau$$

it suffices to establish

$$[\![t_1]\!] = [\![t_2]\!] \in [\![\tau]\!]$$

The power of denotational semantics

Proof principle: to show

$$t_1 \cong_{\mathrm{ctx}} t_2 : \tau$$

it suffices to establish

$$[\![t_1]\!] = [\![t_2]\!] \in [\![\tau]\!]$$

$$\begin{aligned}
\mathcal{C}[t_1] \Downarrow_{\mathtt{nat}} v &\Rightarrow [\![\mathcal{C}[t_1]]\!] = [\![v]\!] && \text{(soundness)}\\
&\Rightarrow [\![\mathcal{C}[t_2]]\!] = [\![v]\!] && \text{(compositionality on } [\![t_1]\!] = [\![t_2]\!]\text{)}\\
&\Rightarrow \mathcal{C}[t_2] \Downarrow_{\mathtt{nat}} v && \text{(adequacy)}
\end{aligned}$$

Proof principle: to show

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\mathcal{C}[t_1] \Downarrow_{\mathtt{nat}} v &\Rightarrow [\![\mathcal{C}[t_1]]\!] = [\![v]\!] && \text{(soundness)} \\
&\Rightarrow [\![\mathcal{C}[t_2]]\!] = [\![v]\!] && \text{(compositionality on } [\![t_1]\!] = [\![t_2]\!]\text{)} \\
&\Rightarrow \mathcal{C}[t_2] \Downarrow_{\mathtt{nat}} v && \text{(adequacy)}
\end{aligned}$$

and symmetrically for $\mathcal{C}[t_2] \Downarrow_{\mathtt{nat}} v \Rightarrow \mathcal{C}[t_1] \Downarrow_{\mathtt{nat}} v$, and similarly for `bool`.

Proof principle: to show

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it suffices to establish

$$[\![t_1]\!] = [\![t_2]\!] \in [\![\tau]\!]$$

Denotational equality is **sound**, but is it **complete**?
Does equality in the model imply contextual equivalence?

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Denotational equality is **sound**, but is it **complete**?
Does equality in the model imply contextual equivalence?

Full abstraction.

Denotational Semantics for PCF

Denotational Semantics for PCF

Types and contexts

Semantics of types

$$[\![\texttt{nat}]\!] \stackrel{\text{def}}{=} \mathbb{N}_\perp \qquad \text{(flat domain)}$$

$$[\![\texttt{bool}]\!] \stackrel{\text{def}}{=} \mathbb{B}_\perp \qquad \text{(flat domain)}$$

$$[\![\tau \to \tau']\!] \stackrel{\text{def}}{=} [\![\tau]\!] \to [\![\tau']\!] \qquad \text{(function domain)}$$

Semantics of contexts

$$[\![\Gamma]\!] \stackrel{\text{def}}{=} \prod_{x \in \mathrm{dom}(\Gamma)} [\![\Gamma(x)]\!] \qquad (\Gamma\text{-environments})$$

Semantics of contexts

$$[\![\Gamma]\!] \stackrel{\text{def}}{=} \prod_{x \in \mathrm{dom}(\Gamma)} [\![\Gamma(x)]\!] \qquad (\Gamma\text{-environments})$$

- $[\![\cdot]\!] = \mathbb{1}$ (one element set)
- $[\![x\colon\tau]\!] = (\{x\} \to [\![\tau]\!]) \cong [\![\tau]\!]$
- $[\![x_1\colon\tau_1, \ldots, x_n\colon\tau_n]\!] = [\![\tau_1]\!] \times \cdots \times [\![\tau_n]\!]$

Denotational Semantics for PCF

Terms

To every typing judgement

$$\Gamma \vdash t : \tau$$

we associate a continuous function

$$[\![\Gamma \vdash t : \tau]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

between domains. In other words,

$$[\![-]\!] : \mathrm{PCF}_{\Gamma,\tau} \to [\![\Gamma]\!] \to [\![\tau]\!]$$

Denotation of operations on $\mathbb{B}$ and $\mathbb{N}$

$$\begin{aligned} \text{succ} : \quad \mathbb{N} &\to \mathbb{N} \\ n &\mapsto n+1 \end{aligned}$$

$$\begin{aligned} \text{pred} : \quad \mathbb{N} &\to \mathbb{N} \\ 0 &\mapsto \text{undefined} \\ n+1 &\mapsto n \end{aligned}$$

$$\begin{aligned} \text{zero?} : \quad \mathbb{N} &\to \mathbb{B} \\ 0 &\mapsto \text{true} \\ n+1 &\mapsto \text{false} \end{aligned}$$

Denotation of operations on $\mathbb{B}$ and $\mathbb{N}$

$$\begin{aligned} \text{succ}_\bot : \quad \mathbb{N}_\bot &\to \mathbb{N}_\bot \\ n &\mapsto n+1 \\ \bot &\mapsto \bot \end{aligned}$$

$$\begin{aligned} \text{pred}_\bot : \quad \mathbb{N}_\bot &\to \mathbb{N}_\bot \\ 0 &\mapsto \bot \\ n+1 &\mapsto n \\ \bot &\mapsto \bot \end{aligned}$$

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Denotation of operations on $\mathbb{B}$ and $\mathbb{N}$

$$
\begin{aligned}
[\![\mathtt{0}]\!](\rho) &\stackrel{\text{def}}{=} 0 && \in \mathbb{N}_\perp \\
[\![\mathtt{true}]\!](\rho) &\stackrel{\text{def}}{=} \text{true} && \in \mathbb{B}_\perp \\
[\![\mathtt{false}]\!](\rho) &\stackrel{\text{def}}{=} \text{false} && \in \mathbb{B}_\perp
\end{aligned}
$$

Denotation of operations on $\mathbb{B}$ and $\mathbb{N}$

$$
\begin{aligned}
[\![\texttt{0}]\!](\rho) &\stackrel{\text{def}}{=} 0 && \in \mathbb{N}_\bot \\
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\end{aligned}
$$

$$
\begin{aligned}
[\![\texttt{succ}(t)]\!](\rho) &\stackrel{\text{def}}{=} \text{succ}_\bot([\![t]\!](\rho)) && \in \mathbb{N}_\bot \\
[\![\texttt{pred}(t)]\!](\rho) &\stackrel{\text{def}}{=} \text{pred}_\bot([\![t]\!](\rho)) && \in \mathbb{N}_\bot \\
[\![\texttt{zero?}(t)]\!](\rho) &\stackrel{\text{def}}{=} \text{zero?}_\bot([\![t]\!](\rho)) && \in \mathbb{B}_\bot
\end{aligned}
$$

$$
[\![\texttt{succ}(t)]\!] = \text{succ}_\bot \circ [\![t]\!]
$$

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\end{aligned}
$$

$$
[\![\mathtt{if}\ b\ \mathtt{then}\ t\ \mathtt{else}\ t']\!] \stackrel{\text{def}}{=} \text{if}([\![b]\!](\rho), [\![t]\!](\rho), [\![t']\!](\rho)) \in [\![\tau]\!]
$$

$$
[\![\mathtt{if}\ b\ \mathtt{then}\ t\ \mathtt{else}\ t']\!] = \text{if} \circ \langle [\![b]\!], \langle [\![t]\!], [\![t']\!] \rangle \rangle
$$

Denotation of the λ-calculus operations

$$[\![x]\!](\rho) \stackrel{\text{def}}{=} \rho(x) \qquad \in [\![\Gamma(x)]\!]$$

$$[\![x]\!](\rho) = \pi_x(\rho)$$

Denotation of the λ-calculus operations

$$\begin{aligned} [\![x]\!](\rho) &\stackrel{\text{def}}{=} \rho(x) && \in [\![\Gamma(x)]\!] \\ [\![t_1\ t_2]\!](\rho) &\stackrel{\text{def}}{=} ([\![t_1]\!](\rho))\ ([\![t_2]\!](\rho)) \end{aligned}$$

$$[\![t_1\ t_2]\!] = \mathrm{eval} \circ \langle [\![t_1]\!], [\![t_2]\!] \rangle$$

Denotation of the λ-calculus operations

$$
\begin{aligned}
[\![x]\!]\,(\rho) &\stackrel{\text{def}}{=} \rho(x) && \in [\![\Gamma(x)]\!] \\
[\![t_1\ t_2]\!]\,(\rho) &\stackrel{\text{def}}{=} ([\![t_1]\!]\,(\rho))\ ([\![t_2]\!]\,(\rho)) \\
[\![\mathtt{fun}\,x{:}\,\tau.\,t]\!]\,(\rho) &\stackrel{\text{def}}{=} \lambda d \in [\![\tau]\!]\,.\ [\![t]\!]\,(\rho, d)
\end{aligned}
$$

$$
[\![\mathtt{fun}\,x{:}\,\tau.\,t]\!] = \mathrm{cur}([\![t]\!])
$$

$$[\![\texttt{fix}\ f]\!](\rho) \stackrel{\text{def}}{=} \mathrm{fix}([\![f]\!](\rho))$$

Denotation of PCF terms

For any PCF term t such that $\Gamma \vdash t : \tau$, the object $[\![t]\!]$ is well-defined and a continuous function $[\![t]\!] : [\![\Gamma]\!] \to \tau$.

For any PCF term t such that $\Gamma \vdash t : \tau$, the object $\llbracket t \rrbracket$
is well-defined and a continuous function $\llbracket t \rrbracket : \llbracket \Gamma \rrbracket \to \tau$.

If $t \in \mathrm{PCF}_\tau$: $\quad \llbracket t \rrbracket \quad \in \quad \llbracket \cdot \rrbracket \to \llbracket \tau \rrbracket \quad = \quad \mathbb{1} \to \llbracket \tau \rrbracket \quad \cong \quad \llbracket \tau \rrbracket$

Denotational Semantics for PCF

Compositionality

Suppose $t, u \in \mathrm{PCF}_{\Gamma,\tau}$, such that

$$[\![t]\!] = [\![u]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

Suppose moreover that $\mathcal{C}[-]$ is a PCF context such that $\Gamma' \vdash_{\Gamma,\tau} \mathcal{C} : \tau'$. Then

$$[\![\mathcal{C}[t]]\!] = [\![\mathcal{C}[u]]\!] : [\![\Gamma']\!] \to [\![\tau']\!] \,.$$

If $\Gamma \vdash_{\Delta,\sigma} \mathcal{C} : \tau$, then define $[\![\mathcal{C}]\!]$ such that

$$[\![\mathcal{C}]\!] : ([\![\Delta]\!] \to [\![\sigma]\!]) \to [\![\Gamma]\!] \to [\![\tau]\!]$$

If $\Gamma \vdash_{\Delta,\sigma} \mathcal{C} : \tau$, then define $[\![\mathcal{C}]\!]$ such that

$$[\![\mathcal{C}]\!] : ([\![\Delta]\!] \to [\![\sigma]\!]) \to [\![\Gamma]\!] \to [\![\tau]\!]$$

$$\begin{aligned} [\![-]\!]\,(d) &= d \\ [\![\mathcal{C}\ t]\!]\,(d)(\rho) &= ([\![\mathcal{C}]\!]\,(d)(\rho))([\![t]\!]\,(\rho)) \\ &\vdots \end{aligned}$$

If $\Gamma \vdash_{\Delta,\sigma} \mathcal{C} : \tau$, then define $[\![\mathcal{C}]\!]$ such that

$$[\![\mathcal{C}]\!] : ([\![\Delta]\!] \to [\![\sigma]\!]) \to [\![\Gamma]\!] \to [\![\tau]\!]$$

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&\ \vdots
\end{aligned}$$

If $\Gamma \vdash_{\Delta,\sigma} \mathcal{C} : \tau$ and $\Delta \vdash t : \sigma$, then

$$[\![\mathcal{C}[t]]\!] = [\![\mathcal{C}]\!]\,([\![t]\!])$$

Assume

$$\Gamma \vdash u : \sigma$$
$$\Gamma, x{:}\sigma \vdash t : \tau$$

Then for all $\rho \in [\![\Gamma]\!]$

$$[\![t[u/x]]\!]\,(\rho) = [\![t]\!]\,(\rho[x \mapsto [\![u]\!]\,(\rho)]).$$

In particular when $\Gamma = \cdot$, $[\![t]\!] : [\![\sigma]\!] \to [\![\tau]\!]$ and

$$[\![t[u/x]]\!] = [\![t]\!]\,([\![u]\!])$$

Denotational Semantics for PCF

Soundness

For all PCF types τ and all closed terms $t, v \in \mathrm{PCF}_\tau$ with v a value, if $t \Downarrow_\tau v$ is derivable, then

$$[\![t]\!] = [\![v]\!] \in [\![\tau]\!]$$

Relating Denotational and Operational Semantics

Reminder: Adequacy

For any closed PCF term t and value v of ground type $\gamma \in \{\mathtt{nat}, \mathtt{bool}\}$

$$[\![t]\!] = [\![v]\!] \in [\![\gamma]\!] \Rightarrow t \Downarrow_{\gamma} v$$

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Adequacy does not hold at function types or for open terms

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Adequacy does **not** hold at function types or for open terms

$$[\![\mathtt{fun}\, x{:}\,\tau.\,(\mathtt{fun}\, y{:}\,\tau.\, y)\, x]\!] \quad = \quad [\![\mathtt{fun}\, x{:}\,\tau.\, x]\!] \quad : [\![\tau]\!] \to [\![\tau]\!]$$

but

$$\mathtt{fun}\, x{:}\,\tau.\,(\mathtt{fun}\, y{:}\,\tau.\, y)\, x \not\Downarrow_{\tau\to\tau} \mathtt{fun}\, x{:}\,\tau.\, x$$

Relating Denotational and Operational Semantics

Formal approximation relation

How to prove adequacy

Proof idea: introduce a relation R such that

1. if $t \in \mathrm{PCF}_{\texttt{nat}}$, $n \in \mathbb{N}$, and $R(n, t)$, then $t \Downarrow_{\gamma} \underline{n}$ (same for booleans);
2. for any well-typed term t, $R(\llbracket t \rrbracket, t)$;

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Thus $v = \underline{n}$ for some $n \in \mathbb{N}$, and $\llbracket v \rrbracket = n$.

$$\begin{aligned} \llbracket t \rrbracket = \llbracket \underline{n} \rrbracket &= n \\ &\Rightarrow R(n, t) \\ &\Rightarrow t \Downarrow \underline{n} = v \end{aligned}$$

Proof idea: introduce a relation R such that

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But at non-base types, adequacy does not hold.

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But at non-base types, adequacy does not hold.

We must define a **family** of relations, tailored for each type: formal approximation

$$\lhd_\tau \subseteq \llbracket \tau \rrbracket \times \mathrm{PCF}_\tau$$

Formal approximation at base types

$$d \lhd_{\mathtt{nat}} t \overset{\text{def}}{\Leftrightarrow} (d \in \mathbb{N} \Rightarrow t \Downarrow_{\mathtt{nat}} \underline{d})$$

$$\begin{aligned} d \lhd_{\mathtt{bool}} t \overset{\text{def}}{\Leftrightarrow} & (d = \text{true} \Rightarrow t \Downarrow_{\mathtt{bool}} \mathtt{true}) \\ & \wedge (d = \text{false} \Rightarrow t \Downarrow_{\mathtt{bool}} \mathtt{false}) \end{aligned}$$

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Exactly what we need to get 1.

Formal approximation at base types

$$d \lhd_{\texttt{nat}} t \overset{\text{def}}{\Leftrightarrow} (d \in \mathbb{N} \Rightarrow t \Downarrow_{\texttt{nat}} \underline{d})$$

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Exactly what we need to get 1.

Note though that $\bot \lhd_{\texttt{nat}} t$ for any $t \in \mathrm{PCF}_{\texttt{nat}}$.

Formal approximation at function types

1. if $t \in \mathrm{PCF}_{\mathtt{nat}}$, $n \in \mathbb{N}$, and $R(n, t)$, then $t \Downarrow_{\gamma} \underline{n}$ (same for booleans); ✓
2. for any well-typed term t, $R(\llbracket t \rrbracket, t)$.

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Assume $\llbracket u \rrbracket \lhd_\tau u$ and $\llbracket t \rrbracket \lhd_{\tau \to \tau'} t$, how do we get $\llbracket t\ u \rrbracket = \llbracket t \rrbracket (\llbracket u \rrbracket) \lhd_\tau t\ u$?

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Assume $[\![u]\!] \lhd_\tau u$ and $[\![t]\!] \lhd_{\tau \to \tau'} t$, how do we get $[\![t\ u]\!] = [\![t]\!]([\![u]\!]) \lhd_\tau t\ u$?

Define

$$d \lhd_{\tau \to \tau'} t \overset{\text{def}}{\Leftrightarrow} \forall e \in [\![\tau]\!], u \in \mathrm{PCF}_\tau . (e \lhd_\tau u \Rightarrow d(e) \lhd_{\tau'} t\ u)$$

Formal approximation for open terms

$$\textsc{Abs}\ \frac{\Gamma, x{:}\tau \vdash t : \tau'}{\Gamma \vdash \mathtt{fun}\, x{:}\tau.\, t : \tau \to \tau'}$$

To prove Item 2, we need to talk about open terms.

$$\textsc{Abs}\ \frac{\Gamma, x{:}\tau \vdash t : \tau'}{\Gamma \vdash \mathsf{fun}\, x{:}\tau.\, t : \tau \to \tau'}$$

To prove Item 2, we need to talk about **open** terms.

$$[\![t]\!]\,([\![u]\!]) = [\![(t[u/x])]\!]$$

Semantic application ≈ syntactic substitution

$$\text{ABS}\ \frac{\Gamma, x{:}\tau \vdash t : \tau'}{\Gamma \vdash \mathsf{fun}\, x{:}\tau.\, t : \tau \to \tau'}$$

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$$[\![t]\!]\,([\![u]\!]) = [\![(t[u/x])]\!]$$ Semantic application ≈ syntactic substitution

Fundamental property of formal approximation

Given a term t such that $\Gamma \vdash t : \tau$ for some Γ and τ, for any environment ρ and substitution σ such that $\rho \lhd_\Gamma \sigma$, we have $[\![t]\!]\,(\rho) \lhd_\tau t[\sigma]$.

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Fundamental property of formal approximation

Given a term t such that $\Gamma \vdash t : \tau$ for some Γ and τ, for any environment ρ and **substitution** σ such that $\rho \lhd_\Gamma \sigma$, we have $[\![t]\!]\,(\rho) \lhd_\tau t[\sigma]$.

Parallel substitution: maps each $x \in \mathrm{dom}(\Gamma)$ to $\sigma(x) \in \mathrm{PCF}_{\Gamma(x)}$.

Relating Denotational and Operational Semantics

Proof of the fundamental property of formal approximation

Properties of formal approximation

1. The least element approximates any program: for any τ and $t \in \mathrm{PCF}_\tau$, $\bot_{[\![\tau]\!]} \lhd_\tau t$;

2. the set $\{d \in [\![\tau]\!] \mid d \lhd_\tau t\}$ is chain-closed;

1. The least element approximates any program: for any τ and $t \in \mathrm{PCF}_\tau$, $\perp_{[\![\tau]\!]} \lhd_\tau t$;

2. the set $\{d \in [\![\tau]\!] \mid d \lhd_\tau t\}$ is chain-closed;

3. if $\forall v.\, t \Downarrow_\tau v \Rightarrow t' \Downarrow_\tau v$, and $d \lhd_\tau t$, then $d \lhd_\tau t'$.

Relating Denotational and Operational Semantics

Extensionality

Characterizing formal approximation

Contextual preorder is the one-sided version of contextual equivalence: $\Gamma \vdash t \leq_{\mathrm{ctx}} t' : \tau$ if for all $\mathcal{C}$ such that $\cdot \vdash_{\Gamma,\tau} \mathcal{C} : \gamma$ and for all values v,

$$\mathcal{C}[t] \Downarrow_{\gamma} v \Rightarrow \mathcal{C}[t'] \Downarrow_{\gamma} v.$$

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$$\mathcal{C}[t] \Downarrow_\gamma v \Rightarrow \mathcal{C}[t'] \Downarrow_\gamma v.$$

$$\Gamma \vdash t \cong_{\text{ctx}} t' : \tau \Leftrightarrow (\Gamma \vdash t \leq_{\text{ctx}} t' : \tau \wedge \Gamma \vdash t' \leq_{\text{ctx}} t : \tau)$$

Contextual preorder is the one-sided version of contextual equivalence: $\Gamma \vdash t \leq_{\mathrm{ctx}} t' : \tau$ if for all $\mathcal{C}$ such that $\cdot \vdash_{\Gamma,\tau} \mathcal{C} : \gamma$ and for all values v,

$$\mathcal{C}[t] \Downarrow_\gamma v \Rightarrow \mathcal{C}[t'] \Downarrow_\gamma v.$$

$$\Gamma \vdash t \cong_{\mathrm{ctx}} t' : \tau \Leftrightarrow (\Gamma \vdash t \leq_{\mathrm{ctx}} t' : \tau \wedge \Gamma \vdash t' \leq_{\mathrm{ctx}} t : \tau)$$

It **corresponds to formal approximation**: for all PCF types τ and closed terms $t_1, t_2 \in \mathrm{PCF}_\tau$

$$t_1 \leq_{\mathrm{ctx}} t_2 : \tau \Leftrightarrow [\![t_1]\!] \lhd_\tau t_2.$$

For contextual preorder between closed terms, applicative contexts are enough.

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Let t_1, t_2 be closed terms of type τ. Then $t_1 \leq_{\text{ctx}} t_2 : \tau$ if and only if, for every term $f : \tau \to \texttt{bool}$,

$$f\ t_1 \Downarrow_{\texttt{bool}} \texttt{true} \Rightarrow f\ t_2 \Downarrow_{\texttt{bool}} \texttt{true}.$$

Extensionality properties of contextual preorder

For $\gamma = \texttt{bool}$ or $\texttt{nat}$, $t_1 \leq_{\text{ctx}} t_2 : \tau$ holds if and only if

$$\forall v.\, (t_1 \Downarrow_\gamma v \Rightarrow t_2 \Downarrow_\gamma v).$$

For $\gamma = \texttt{bool}$ or $\texttt{nat}$, $t_1 \leq_{\mathrm{ctx}} t_2 : \tau$ holds if and only if

$$\forall v.\, (t_1 \Downarrow_\gamma v \Rightarrow t_2 \Downarrow_\gamma v).$$

At a function type $\tau \to \tau'$, $t_1 \leq_{\mathrm{ctx}} t_2 : \tau \to \tau'$ holds if and only if

$$\forall t \in \mathrm{PCF}_\tau \,.\, (t_1\, t \leq_{\mathrm{ctx}} t_2\, t : \tau').$$

Full abstraction

Full abstraction

Failure of full abstraction

Full abstraction

A denotational model is **fully abstract** if

$$t_1 \cong_{\text{ctx}} t_2 : \tau \Rightarrow [\![t_1]\!] = [\![t_2]\!] \in [\![\tau]\!]$$

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A form of **completeness** of semantic equivalence wrt. program equivalence.

Full abstraction

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A form of completeness of semantic equivalence wrt. program equivalence.

The domain model of PCF is *not* fully abstract.

The *parallel or* function $\mathbf{por} : \mathbb{B}_\perp \times \mathbb{B}_\perp \to \mathbb{B}_\perp$ is defined as given by the following table:

por	true	false	$\perp$
true	true	true	true
false	true	false	$\perp$
$\perp$	true	$\perp$	$\perp$

The (left) sequential or function $\mathbf{or} : \mathbb{B}_\perp \times \mathbb{B}_\perp \to \mathbb{B}_\perp$ is defined as

$$\mathrm{or} \stackrel{\mathrm{def}}{=} [\![\mathtt{fun}\, x \colon \mathtt{bool}.\ \mathtt{fun}\, y \colon \mathtt{bool}.\ \mathtt{if}\ x\ \mathtt{then}\ \mathtt{true}\ \mathtt{else}\ y]\!]$$

It is given by the following table:

or	true	false	$\perp$
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false	true	false	$\perp$
$\perp$	$\perp$	$\perp$	$\perp$

Parallel vs sequential or

por	true	false	$\bot$
true	true	true	true
false	true	false	$\bot$
$\bot$	true	$\bot$	$\bot$

or	true	false	$\bot$
true	true	true	true
false	true	false	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$

por	true	false	$\bot$
true	true	true	true
false	true	false	$\bot$
$\bot$	true	$\bot$	$\bot$

or	true	false	$\bot$
true	true	true	true
false	true	false	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$

or is sequential, but por is not.

Undefinability or parallel or

There is **no** closed PCF term

$$t : \texttt{bool} \to \texttt{bool} \to \texttt{bool}$$

satisfying

$$[\![t]\!] = \mathrm{por} : \mathbb{B}_\perp \to \mathbb{B}_\perp \to \mathbb{B}_\perp \ .$$

Failure of full abstraction

The denotational model of PCF in domains and continuous functions is not fully abstract.

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For well-chosen $T_{\texttt{true}}$ and $T_{\texttt{false}}$,

$$T_{\texttt{true}} \cong_{\text{ctx}} T_{\texttt{false}} : (\texttt{bool} \to \texttt{bool} \to \texttt{bool}) \to \texttt{bool}$$

$$[\![T_{\texttt{true}}]\!] \neq [\![T_{\texttt{false}}]\!] \in (\mathbb{B} \to \mathbb{B} \to \mathbb{B}) \to \mathbb{B}$$

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Idea:

- for all $f \in PCF_{\mathtt{bool}\to\mathtt{bool}\to\mathtt{bool}}$, ensure $T_b\ f \Uparrow_{\mathtt{bool}}$...
- but $[\![T_b]\!]\ (\mathrm{por}) = [\![b]\!]$.

Example of full abstraction failure

$$
\begin{aligned}
T_b \stackrel{\text{def}}{=}\ & \texttt{fun}\ f\colon \texttt{bool} \to (\texttt{bool} \to \texttt{bool}).\\
&\quad \texttt{if}(f\ \texttt{true}\ \Omega_{\texttt{bool}})\ \texttt{then}\\
&\qquad \texttt{if}\ (f\ \Omega_{\texttt{bool}}\ \texttt{true})\ \texttt{then}\\
&\qquad\quad \texttt{if}\ (f\ \texttt{false}\ \texttt{false})\ \texttt{then}\ \Omega_{\texttt{bool}}\ \texttt{else}\ b\\
&\qquad \texttt{else}\ \Omega_{\texttt{bool}}\\
&\quad \texttt{else}\ \Omega_{\texttt{bool}}
\end{aligned}
$$

Full abstraction

Beyond full abstraction failure

Interpreting full abstraction failure

- PCF is not expressive enough to present the model?
- The model does not adequately capture PCF?
- Contexts are too weak: they do not distinguish enough programs?

PCF+por

$\boxed{\Gamma \vdash t : \tau}$

$$\dots \qquad \textsc{Por}\ \frac{\Gamma \vdash t_1 : \tau \qquad \Gamma \vdash t_2 : \tau}{\Gamma \vdash \mathtt{por}(t_1, t_2) : \tau}$$

$\boxed{t \Downarrow_\tau v}$

$$\textsc{PorL}\ \frac{t_1 \Downarrow_{\mathtt{bool}} \mathtt{true}}{\mathtt{por}(t_1, t_2) \Downarrow_{\mathtt{bool}} \mathtt{true}} \qquad \textsc{PorR}\ \frac{t_2 \Downarrow_{\mathtt{bool}} \mathtt{true}}{\mathtt{por}(t_1, t_2) \Downarrow_{\mathtt{bool}} \mathtt{true}}$$

$$\textsc{PorF}\ \frac{t_1 \Downarrow_{\mathtt{bool}} \mathtt{false} \qquad t_2 \Downarrow_{\mathtt{bool}} \mathtt{false}}{\mathtt{por}(t_1, t_2) \Downarrow_{\mathtt{bool}} \mathtt{false}}$$

Full abstraction for PCF+`por`

If we extend the semantics of PCF to PCF+`por` with

$$[\![\texttt{por}]\!] = \mathrm{por}$$

the resulting denotational semantics is fully abstract.

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the resulting denotational semantics is fully abstract...

but is PCF+`por` still a reasonable model of programming language?

Fully abstract semantics for PCF

- first step: dI-domains & stable functions $\rightarrow$ no $\mathbf{por}$ any more, but still not fully abstract...
- only proper answers in the late 90s (!): logical relations and game semantics

Fully abstract semantics

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Real languages have effects

- If you add effects (references, control flow...) to a language, contexts become *much more* expressive.
- Full abstraction becomes different: somewhat easier... but is contextual equivalence still a reasonable idea?

Where to go from here?

Towards full abstraction

Source of a very rich literature:

- linear logic
- logical relations
- game semantics
- bisimulations techniques
- ...

Separate

- the structure needed to interpret a language (generic)
- how to construct this structure in particular examples (specific)

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Interpret:

- a type τ as an object in a category;
- a term $\Gamma \vdash t : \tau$ as a morphism/arrow $[\![t]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$.

Categorical semantics

Separate

- the structure needed to interpret a language (generic)
- how to construct this structure in particular examples (specific)

Interpret:

- a type τ as an object in a category;
- a term $\Gamma \vdash t : \tau$ as a morphism/arrow $[\![t]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$.

Example: λ-calculus → cartesian closed categories

OCaml's ADT:

It is a fixed point equation! We can use domain theory to solve it.

Beyond pure languages

Effects: control flow (errors), mutability/state, input-output...
An important aspect of programming languages!

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Beyond pure languages

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An important aspect of programming languages!

Modelled as a monad T (example: $T(A) \stackrel{\text{def}}{=} (A \times \mathrm{State})^{\mathrm{State}}$)

Denotation of a computation: $[\![\Gamma]\!] \to T([\![\tau]\!])$

More semantics

Easter: **axiomatic semantic** (Hoare Logic and Model Checking)

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In the end, the most interesting aspects of semantics is in the interaction between different approaches.