Compositionality:  $\llbracket t \rrbracket = \llbracket t' \rrbracket \Rightarrow \llbracket C[t] \rrbracket = \llbracket C[t'] \rrbracket$ . Soundness: for any type  $\tau$ ,  $t \downarrow_{\tau} v \Rightarrow \llbracket t \rrbracket = \llbracket v \rrbracket$ . Adequacy: for  $\gamma = \text{bool}$  or nat, if  $t \in \text{PcF}_{\gamma}$  and  $\llbracket t \rrbracket = \llbracket v \rrbracket$  then  $t \downarrow_{\gamma} v$ . Compositionality:  $\llbracket t \rrbracket = \llbracket t' \rrbracket \Rightarrow \llbracket C[t] \rrbracket = \llbracket C[t'] \rrbracket$ . Soundness: for any type  $\tau$ ,  $t \downarrow_{\tau} v \Rightarrow \llbracket t \rrbracket = \llbracket v \rrbracket$ . Adequacy: for  $\gamma = \text{bool}$  or nat, if  $t \in \text{PcF}_{\gamma}$  and  $\llbracket t \rrbracket = \llbracket v \rrbracket$  then  $t \downarrow_{\gamma} v$ .

Now: back to contextual equivalence...

ADEQUACY Extensionality **Contextual preorder** is the one-sided version of contextual equivalence:  $\Gamma \vdash t \leq_{\text{ctx}} t' : \tau$  if for all C such that  $\cdot \vdash_{\Gamma,\tau} C : \gamma$  and for all values  $\nu$ ,

 $\mathcal{C}[t] \Downarrow_{\gamma} \nu \Rightarrow \mathcal{C}[t'] \Downarrow_{\gamma} \nu.$ 

**Contextual preorder** is the one-sided version of contextual equivalence:  $\Gamma \vdash t \leq_{\text{ctx}} t' : \tau$  if for all C such that  $\cdot \vdash_{\Gamma,\tau} C : \gamma$  and for all values  $\nu$ ,

$$\begin{array}{ccc} C[t] \Downarrow_{\gamma} v \Rightarrow C[t'] \Downarrow_{\gamma} v. \\ \gamma = & \text{true on false} \\ \varphi & \text{not} & 0 & \underline{\mathcal{M}} \end{array}$$

 $\Gamma \vdash t \cong_{\mathsf{ctx}} t' : \tau \Leftrightarrow (\Gamma \vdash t \leq_{\mathsf{ctx}} t' : \tau \land \Gamma \vdash t' \leq_{\mathsf{ctx}} t : \tau)$ 

Let au be a type, and assume  $t_1, t_2 \in \mathsf{PCF}_{ au}$  are such that  $t_1 \leq_{\mathrm{ctx}} t_2 : au$ . Then

By induction on T: mat: Take G = - then (Et1)=t1 GTt1)=t2 Bly cont. prealer: ([t] UN=> ([t] UN t', (2) By the versions emma date => d x may te bool: nimila

Tot: Ned if dant, and antit Take e E [[T], MERGE: NT. equ Assumption: de azity u By IN out it is enough to show the suder the to get de ten.  $\mathcal{E}[t_{y}u] = \mathcal{E}'[t_{x}] \text{ for } \mathcal{E}' = \mathcal{E}[-u]$ and no  $t_{x} \leq c_{x}v_{x}$  thue  $t_{y}u \leq c_{x}v_{x}$ 

To characterise contextual preorder between closed terms, **applicative** contexts are enough.

To characterise contextual preorder between closed terms, applicative contexts are enough.

Let  $t_1, t_2$  be closed terms of type  $\tau$ . Then  $t_1 \leq_{\mathrm{ctx}} t_2 : \tau$  if and only if, for every term  $f : \tau \to \mathsf{bool}$ ,

 $f t_1 \downarrow_{\text{bool}} \text{true} \Rightarrow f t_2 \downarrow_{\text{bool}} \text{true}.$ 

 $\exists if t_1 \leq t_2 \quad Hen (i - )[t_1] \Rightarrow (i - )[t_1]$ It, Vo It, Vo El Assume UP EPCF , pool, Stalt time aft. Utime? let be a context: . F. , t G: y with & shop Define f := fern a: T. C[2] We want GEtz] W, v » GEtz] W, v Define + g, v: X » bool x. g u W true es u W v

Apply lossimption with good (good)(t) I true = (good)(t) I true et, Vo P.St. Vo CITY WV

Formal approximation corresponds to the contextual preorder.

Formal approximation corresponds to the contextual preorder.

For all PCF types au and all closed terms  $t_1, t_2 \in \mathsf{PCF}_{ au}$ 

 $t_1 \leq_{\mathrm{ctx}} t_2 : \tau \Leftrightarrow \llbracket t_1 \rrbracket \triangleleft_{\tau} t_2.$ 

I tosume the scrite Dy pendemental theorem: It, D<12t1 By monstanicity: It, D<22t2 El Assume Itji) 47 te It is change to show that for all f: T-> bool of to ytime them of to y true Dy the fundamental lemma of 2 - T-> hood of So TID (TEA) Close ft.

(Tgts) <br/>
top ft,<br/>
gts the state of the (soundness)<br/>
=) gte it true (def of those)

For  $\gamma = \texttt{bool}$  or nat,  $t_1 \leq_{\texttt{ctx}} t_2 : \gamma$  holds if and only if

 $\forall v. (t_1 \Downarrow_{\gamma} v \Rightarrow t_2 \Downarrow_{\gamma} v).$ 

For  $\gamma = \text{bool or nat}, t_1 \leq_{\text{ctx}} t_2 : \gamma$  holds if and only if

 $\forall \nu. (t_1 \Downarrow_{\gamma} \nu \Rightarrow t_2 \Downarrow_{\gamma} \nu).$ 

At a function type  $\tau \to \tau'$ ,  $t_1 \leq_{\text{ctx}} t_2 : \tau \to \tau'$  holds if and only if  $\forall t \in \mathsf{PCF}_{\tau} . (t_1 t \leq_{\text{ctx}} t_2 t : \tau').$ 

### FULL ABSTRACTION

# FULL ABSTRACTION

FAILURE OF FULL ABSTRACTION

#### A denotational model is **fully abstract** if

$$t_1 \cong_{\mathsf{ctx}} t_2 : \tau \Rightarrow \llbracket t_1 \rrbracket = \llbracket t_2 \rrbracket \in \llbracket \tau \rrbracket$$

#### A denotational model is **fully abstract** if

$$t_1 \cong_{\mathrm{ctx}} t_2 : \tau \Longrightarrow \llbracket t_1 \rrbracket = \llbracket t_2 \rrbracket \in \llbracket \tau \rrbracket$$

#### A form of **completeness** of semantic equivalence wrt. program equivalence.

#### A denotational model is **fully abstract** if

$$t_1 \cong_{\mathrm{ctx}} t_2 : \tau \Longrightarrow \llbracket t_1 \rrbracket = \llbracket t_2 \rrbracket \in \llbracket \tau \rrbracket$$

A form of completeness of semantic equivalence wrt. program equivalence.

## The domain model of PCF is **not** fully abstract.

The parallel or function  $\text{por} : \mathbb{B}_{\perp} \times \mathbb{B}_{\perp} \to \mathbb{B}_{\perp}$  is defined as given by the following table:

por	true	false	$\perp$
true	true	true	true
false	true	false	$\perp$
$\perp$	true	$\perp$	$\bot$

The (left) sequential or function  $or: \mathbb{B}_\perp \times \mathbb{B}_\perp \to \mathbb{B}_\perp$  is defined as

or  $\stackrel{\text{def}}{=} \llbracket \text{fun } x : \text{bool. fun } y : \text{bool. if } x \text{ then true else } y \rrbracket$ 

It is given by the following table:

or	true	false	$\perp$
true	true	true	true
false	true	false	$\perp$
$\perp$	$\perp$	$\perp$	$\bot$

por	true	false	$\perp$	or	true	false	$\perp$
true	true	true	true	true	true	true	true
false	true	false	$\perp$	false	true	false	$\perp$
$\perp$	true	$\perp$	$\perp$	$\perp$	T	$\perp$	$\perp$

por	true	false	$\perp$	or	true	false	$\perp$
true	true	true	true	true	true	true	true
false	true	false	$\perp$	false	true	false	$\perp$
$\perp$	true	$\perp$	$\perp$	$\perp$	T	$\perp$	$\perp$

or is sequential, but por is not.

#### There is **no** closed PCF term

 $t: bool \rightarrow bool \rightarrow bool$ 

satisfying

$$\llbracket t \rrbracket = \operatorname{por} : \mathbb{B}_{\perp} \to \mathbb{B}_{\perp} \to \mathbb{B}_{\perp}$$
.

The denotational model of PCF in domains and continuous functions is not fully abstract.

The denotational model of PCF in domains and continuous functions is not fully abstract.

For well-chosen  $T_{true}$  and  $T_{false}$ ,

$$T_{\text{true}} \cong_{\text{ctx}} T_{\text{false}} : (\text{bool} \to \text{bool} \to \text{bool}) \to \text{bool}$$
$$[\![T_{\text{true}}]\!] \neq [\![T_{\text{false}}]\!] \in (\mathbb{B} \to \mathbb{B} \to \mathbb{B}) \to \mathbb{B}$$

The denotational model of PCF in domains and continuous functions is not fully abstract.

For well-chosen  $T_{true}$  and  $T_{false}$ ,

$$T_{\text{true}} \cong_{\text{ctx}} T_{\text{false}} : (\text{bool} \to \text{bool} \to \text{bool}) \to \text{bool}$$
$$[[T_{\text{true}}]] \neq [[T_{\text{false}}]] \in (\mathbb{B} \to \mathbb{B} \to \mathbb{B}) \to \mathbb{B}$$

Idea:

- for all  $f \in PCF_{bool \rightarrow bool \rightarrow bool}$ , ensure  $T_b f \uparrow_{bool}$ ...
- but  $\llbracket T_b \rrbracket$  (por) =  $\llbracket b \rrbracket$ .

```
 \begin{array}{l} T_b \stackrel{\mathrm{def}}{=} & \mathrm{fun}\,f{:}\,\mathrm{bool}\to(\mathrm{bool}\to\mathrm{bool}).\\ & \mathrm{if}(f\,\mathrm{true}\,\Omega_{\mathrm{bool}})\,\mathrm{then}\\ & \mathrm{if}\,(f\,\Omega_{\mathrm{bool}}\,\mathrm{true})\,\mathrm{then}\\ & \mathrm{if}\,(f\,\mathrm{false}\,\mathrm{false})\,\mathrm{then}\,\Omega_{\mathrm{bool}}\,\mathrm{else}\,b\\ & \mathrm{else}\,\Omega_{\mathrm{bool}}\\ & \mathrm{else}\,\Omega_{\mathrm{bool}} \end{array}
```

TTb1) (por) = 76 1

 $\checkmark$