Where we're at

We have a denotational semantics for types $[\![\tau]\!]$ and terms $[\![t]\!]$ such that:

Compositionality: $[\![t]\!] = [\![t']\!] \Rightarrow [\![\mathcal{C}[t]]\!] = [\![\mathcal{C}[t']]\!]$. ✓

Soundness: for any type τ, $t \Downarrow_\tau v \Rightarrow [\![t]\!] = [\![v]\!]$. ✓

Adequacy: for $\gamma = \texttt{bool}$ or $\texttt{nat}$, if $t \in \mathrm{PCF}_\gamma$ and $[\![t]\!] = [\![v]\!]$ then $t \Downarrow_\gamma v$. ✓

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From this we can show

$$[\![t]\!] = [\![u]\!] \in [\![\tau]\!] \Rightarrow t \cong_{\mathrm{ctx}} u : \tau$$

What about the converse implication?

Full abstraction

Full abstraction

Failure of full abstraction

Full abstraction

A denotational model is **fully abstract** if

$$t_1 \cong_{\mathrm{ctx}} t_2 : \tau \Rightarrow [\![t_1]\!] = [\![t_2]\!] \in [\![\tau]\!]$$

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A form of completeness of semantic equivalence wrt. program equivalence.

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A form of **completeness** of semantic equivalence wrt. program equivalence.

The domain model of PCF is *not* fully abstract.

Parallel or

The *parallel or* function $\mathbf{por} : \mathbb{B}_\perp \times \mathbb{B}_\perp \to \mathbb{B}_\perp$ is defined as given by the following table:

por	true	false	$\perp$
true	true	true	true
false	true	false	$\perp$
$\perp$	true	$\perp$	$\perp$

The (left) sequential or function $\mathbf{or} : \mathbb{B}_\perp \times \mathbb{B}_\perp \to \mathbb{B}_\perp$ is defined as

$$\mathrm{or} \stackrel{\mathrm{def}}{=} [\![\texttt{fun}\ x\colon \texttt{bool}.\ \texttt{fun}\ y\colon \texttt{bool}.\ \texttt{if}\ x\ \texttt{then}\ \texttt{true}\ \texttt{else}\ y]\!]$$

It is given by the following table:

or	true	false	$\perp$
true	true	true	true
false	true	false	$\perp$
$\perp$	$\perp$	$\perp$	$\perp$

Parallel vs sequential or

por	true	false	⊥
true	true	true	true
false	true	false	⊥
⊥	true	⊥	⊥

or	true	false	⊥
true	true	true	true
false	true	false	⊥
⊥	⊥	⊥	⊥

Parallel vs sequential or

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true	true	true	true
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⊥	true	⊥	⊥

or	true	false	⊥
true	true	true	true
false	true	false	⊥
⊥	⊥	⊥	⊥

or is sequential, but por is not.

Undefinability or parallel or

There is no closed PCF term

$$t : \texttt{bool} \to \texttt{bool} \to \texttt{bool}$$

satisfying

$$[\![t]\!] = \mathrm{por} : \mathbb{B}_\perp \to \mathbb{B}_\perp \to \mathbb{B}_\perp \ .$$

Failure of full abstraction

The denotational model of PCF in domains and continuous functions is not fully abstract.

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For well-chosen $T_{\texttt{true}}$ and $T_{\texttt{false}}$,

$$T_{\texttt{true}} \cong_{\text{ctx}} T_{\texttt{false}} : (\texttt{bool} \to \texttt{bool} \to \texttt{bool}) \to \texttt{bool}$$

$$[\![T_{\texttt{true}}]\!] \neq [\![T_{\texttt{false}}]\!] \in (\mathbb{B} \to \mathbb{B} \to \mathbb{B}) \to \mathbb{B}$$

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Idea:

- for all $f \in PCF_{\texttt{bool}\to\texttt{bool}\to\texttt{bool}}$, ensure $T_b\ f \Uparrow_{\texttt{bool}}$...
- but $[\![T_b]\!]\ (\text{por}) = [\![b]\!]$.

$$
\begin{aligned}
T_b \stackrel{\text{def}}{=}\ & \texttt{fun}\ f\colon \texttt{bool} \to (\texttt{bool} \to \texttt{bool}).\\
& \quad \texttt{if}(f\ \texttt{true}\ \Omega_{\texttt{bool}})\ \texttt{then}\\
& \qquad \texttt{if}\ (f\ \Omega_{\texttt{bool}}\ \texttt{true})\ \texttt{then}\\
& \qquad\quad \texttt{if}\ (f\ \texttt{false}\ \texttt{false})\ \texttt{then}\ \Omega_{\texttt{bool}}\ \texttt{else}\ b\\
& \qquad \texttt{else}\ \Omega_{\texttt{bool}}\\
& \quad \texttt{else}\ \Omega_{\texttt{bool}}
\end{aligned}
$$

1) $T_b f \Uparrow_{bool}$ for all $f \in PCF_{bool \to bool \to bool}$

$T_b f \Downarrow_{bool} v$ iff

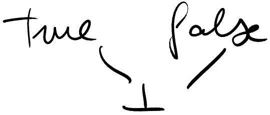
$$\left.\begin{array}{l} f \ \mathsf{true}\ \Omega_{bool} \Downarrow \mathsf{true} \\ f\ \Omega_{bool}\ \mathsf{true} \Downarrow \mathsf{true} \\ f\ \mathsf{false}\ \mathsf{false} \Downarrow \mathsf{false} \end{array}\right\} (1)$$

if f satisfies (1) then

$$\begin{array}{l} [\![f]\!](\mathsf{true}, \bot_B) = \mathsf{true} \\ [\![f]\!](\bot_B, \mathsf{true}) = \mathsf{true} \\ [\![f]\!](\mathsf{false}, \mathsf{false}) = \mathsf{false} \end{array} \quad (2)$$

⟦f⟧	true	false	⊥
true	true	true	true
false	true	false	⊥
⊥	true	⊥	⊥

(e) ⇒ ⟦f⟧ = por



∄ cannot exist

so for every $f \in PCF_{bool \to bool \to bool}$ $[\![f]\!]_{bool}$

a) Dut $[\![\neg b]\!](\rho\sigma) = \neg [\![b]\!]$

Full abstraction

Beyond full abstraction failure

Interpreting full abstraction failure

- PCF is not expressive enough to present the model?
- The model does not adequately capture PCF?
- Contexts are too weak: they do not distinguish enough programs?

PCF+por

$\boxed{\Gamma \vdash t : \tau}$

$$\cdots \qquad \text{POR}\ \frac{\Gamma \vdash t_1 : \tau \qquad \Gamma \vdash t_2 : \tau}{\Gamma \vdash \mathtt{por}(t_1, t_2) : \tau}$$

$\boxed{t \Downarrow_\tau v}$

$$\text{PORL}\ \frac{t_1 \Downarrow_{\mathtt{bool}} \mathtt{true}}{\mathtt{por}(t_1, t_2) \Downarrow_{\mathtt{bool}} \mathtt{true}} \qquad \text{PORR}\ \frac{t_2 \Downarrow_{\mathtt{bool}} \mathtt{true}}{\mathtt{por}(t_1, t_2) \Downarrow_{\mathtt{bool}} \mathtt{true}}$$

$$\text{PORF}\ \frac{t_1 \Downarrow_{\mathtt{bool}} \mathtt{false} \qquad t_2 \Downarrow_{\mathtt{bool}} \mathtt{false}}{\mathtt{por}(t_1, t_2) \Downarrow_{\mathtt{bool}} \mathtt{false}}$$

Full abstraction for PCF+`por`

If we extend the semantics of PCF to PCF+`por` with

$$[\![\texttt{por}]\!] = \mathrm{por}$$

the resulting denotational semantics is fully abstract.

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the resulting denotational semantics is fully abstract…

but is PCF+`por` still a reasonable model of programming language?

Fully abstract semantics for PCF

- first step: dI-domains & stable functions → no $\mathbf{por}$ any more, but still not fully abstract…
- only proper answers in the late 90s (!): logical relations and game semantics

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Real languages have effects

- If you add effects (references, control flow…) to a language, contexts become *much more* expressive.
- Full abstraction becomes different: somewhat easier… but is contextual equivalence still a reasonable idea?

Where to go from here?

Source of a very rich literature:

- linear logic
- logical relations
- game semantics
- bisimulations techniques
- ...

Separate

- the structure needed to interpret a language (generic)
- how to construct this structure in particular examples (specific)

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Interpret:

- a type τ as an object in a category;
- a term $\Gamma \vdash t : \tau$ as a morphism/arrow $[\![t]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$.

Categorical semantics

Separate

- the structure needed to interpret a language (generic)
- how to construct this structure in particular examples (specific)

Interpret:

- a type τ as an object in a category;
- a term $\Gamma \vdash t : \tau$ as a morphism/arrow $[\![t]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$.

Example: λ-calculus → cartesian closed categories

OCaml's ADT:

```
type 'a tree =
  | Leaf
  | Node of 'a * 'a tree * 'a tree
```

It is a fixed point equation! We can use domain theory to solve it.

Beyond pure languages

Effects: control flow (errors), mutability/state, input-output...
An important aspect of programming languages!

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An important aspect of programming languages!

Modelled as a monad T (example: $T(A) \overset{\text{def}}{=} (A \times \text{State})^{\text{State}}$)

$[\![\Gamma]\!] \to ([\![\tau]\!] \times \text{State})^{\text{State}}$

$\cong$

$[\![\Gamma]\!] \times \text{State} \to [\![\tau]\!] \times \text{State}$

Denotation of a computation: $[\![\Gamma]\!] \to T([\![\tau]\!])$

$[\![\Gamma]\!] \to T([\![\sigma]\!])$

More semantics

Easter: **axiomatic semantic** (Hoare Logic and Model Checking)

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In the end, the most interesting aspects of semantics is in the interaction between different approaches.