Where we're at

We want:

- a mapping of PCF types τ to domains $[\![\tau]\!]$; ✓
- a mapping of closed, well-typed PCF terms $\cdot \vdash t : \tau$ to elements $[\![t]\!] \in [\![\tau]\!]$; ✓
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Such that:

Compositionality: $[\![t]\!] = [\![t']\!] \Rightarrow [\![\mathcal{C}[t]]\!] = [\![\mathcal{C}[t']]\!]$. ✓

Soundness: for any type τ, $t \Downarrow_\tau v \Rightarrow [\![t]\!] = [\![v]\!]$. ✓

Adequacy: for $\gamma = \texttt{bool}$ or $\texttt{nat}$, if $t \in \mathrm{PCF}_\gamma$ and $[\![t]\!] = [\![v]\!]$ then $t \Downarrow_\gamma v$. ✗

Relating Denotational and Operational Semantics

Reminder: Adequacy

For any **closed** PCF term t and value v of **ground** type $\gamma \in \{\texttt{nat}, \texttt{bool}\}$

$$[\![t]\!] = [\![v]\!] \in [\![\gamma]\!] \Rightarrow t \Downarrow_\gamma v$$

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Adequacy does **not** hold at function types or for open terms

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Adequacy does not hold at function types or for open terms

$$[\![\mathtt{fun}\, x{:}\,\tau.\,(\mathtt{fun}\, y{:}\,\tau.\,y)\, x]\!] \quad = \quad [\![\mathtt{fun}\, x{:}\,\tau.\, x]\!] \quad : [\![\tau]\!] \to [\![\tau]\!]$$

but

$$\mathtt{fun}\, x{:}\,\tau.\,(\mathtt{fun}\, y{:}\,\tau.\,y)\, x \not\Downarrow_{\tau\to\tau} \mathtt{fun}\, x{:}\,\tau.\, x$$

Relating Denotational and Operational Semantics

Formal approximation relation

Proof idea: introduce a relation R such that

1. if $t \in \mathrm{PCF}_{\mathtt{nat}}$, $n \in \mathbb{N}$, and $R(n, t)$, then $t \Downarrow_\gamma \underline{n}$ (same for booleans);
2. for any well-typed term t, $R(\llbracket t \rrbracket, t)$;

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$$\begin{aligned} \llbracket t \rrbracket = \left\llbracket \underline{n} \right\rrbracket &= n \\ \Rightarrow R(n, t)& \\ \Rightarrow t \Downarrow \underline{n} &= v \end{aligned}$$

How to prove adequacy

Proof idea: introduce a relation R such that

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But at non-base types, adequacy does not hold.

We must define a **family** of relations, tailored for each type: formal approximation

$$\lhd_\tau \subseteq \llbracket \tau \rrbracket \times \mathrm{PCF}_\tau$$

$$d \lhd_{\mathtt{nat}} t \overset{\text{def}}{\Leftrightarrow} (d \in \mathbb{N} \Rightarrow t \Downarrow_{\mathtt{nat}} \underline{d})$$

$$\begin{aligned} d \lhd_{\mathtt{bool}} t \overset{\text{def}}{\Leftrightarrow} & (d = \mathrm{true} \Rightarrow t \Downarrow_{\mathtt{bool}} \mathtt{true}) \\ & \wedge (d = \mathrm{false} \Rightarrow t \Downarrow_{\mathtt{bool}} \mathtt{false}) \end{aligned}$$

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Exactly what we need to get 1.

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Exactly what we need to get 1.

Note though that $\bot \lhd_{\mathtt{nat}} t$ for any $t \in \mathrm{PCF}_{\mathtt{nat}}$.

Formal approximation at function types

1. if $t \in \mathrm{PCF}_{\mathtt{nat}}$, $n \in \mathbb{N}$, and $R(n, t)$, then $t \Downarrow_\gamma \underline{n}$ (same for booleans); ✓
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Define

$$d \lhd_{\tau \to \tau'} t \overset{\text{def}}{\Leftrightarrow} \forall e \in \llbracket \tau \rrbracket, u \in \mathrm{PCF}_\tau . (e \lhd_\tau u \Rightarrow d(e) \lhd_{\tau'} t\ u)$$

$$\textsc{Abs}\ \frac{\Gamma, x{:}\tau \vdash t : \tau'}{\Gamma \vdash \mathtt{fun}\, x{:}\tau.\, t : \tau \rightarrow \tau'}$$

To prove Item 2, we need to talk about open terms.

$$\text{ABS}\ \frac{\Gamma, x{:}\tau \vdash t : \tau'}{\Gamma \vdash \mathsf{fun}\, x{:}\tau.\, t : \tau \to \tau'}$$

To prove Item 2, we need to talk about **open** terms.

$$[\![t]\!]\,([\![u]\!]) = [\![(t[u/x])]\!] \qquad \text{Semantic application} \approx \text{syntactic substitution}$$

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Fundamental property of formal approximation

Given a term t such that $\Gamma \vdash t : \tau$ for some Γ and τ, for any environment ρ and substitution σ such that $\rho \lhd_\Gamma \sigma$, we have $[\![t]\!]\,(\rho) \lhd_\tau t[\sigma]$.

Formal approximation for open terms

$$\textsc{Abs}\ \frac{\Gamma, x{:}\tau \vdash t : \tau'}{\Gamma \vdash \mathsf{fun}\, x{:}\tau.\, t : \tau \to \tau'}$$

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Given a term t such that $\Gamma \vdash t : \tau$ for some Γ and τ, for any environment ρ and **substitution** σ such that $\rho \lhd_\Gamma \sigma$, we have $[\![t]\!]\,(\rho) \lhd_\tau t[\sigma]$.

Parallel substitution: maps each $x \in \mathbf{dom}(\Gamma)$ to $\sigma(x) \in \mathrm{PCF}_{\Gamma(x)}$.

$\sigma \in \prod_{x \in dom(\Gamma)} PCF_{\Gamma(x)}$

$\rho \in [\![\Gamma]\!] = \prod_{x \in dom(\Gamma)} [\![\Gamma(x)]\!]$

Relating Denotational and Operational Semantics

Proof of the fundamental property of formal approximation

Properties of formal approximation

1. The least element approximates any program: for any τ and $t \in \mathrm{PCF}_\tau$, $\bot_{[\![\tau]\!]} \lhd_\tau t$;

2. the set $\{d \in [\![\tau]\!] \mid d \lhd_\tau t\}$ is chain-closed;

1. for any t τ, $t \in PCF_\tau$ $\quad \bot_{[\![\tau]\!]} \lhd_\tau t$

induct° on τ

nat/bool : $(\underbrace{\bot_{nat} \in \mathbb{N}}_{\text{false}} \Rightarrow \dots)$ vacuously true

$\tau \to \tau'$: we need $\bot_{[\![\tau]\!] \to [\![\tau']\!]} \lhd_{\tau \to \tau'} t \iff \forall d \in [\![\tau]\!], \forall u \in PCF_\tau,$
$d \lhd_\tau u \Rightarrow \underbrace{\bot(d)}_{\bot_{[\![\tau']\!]}} \lhd_{\tau'} t\, u$

IH : $\bot_{[\![\tau']\!]} \lhd_{\tau'} t\, u$

Properties of formal approximation

1. The least element approximates any program: for any τ and $t \in \mathrm{PCF}_\tau$, $\bot_{[\![\tau]\!]} \lhd_\tau t$;

2. the set $\{d \in [\![\tau]\!] \mid d \lhd_\tau t\}$ is chain-closed;

3. if $\forall v.\, t \Downarrow_\tau v \Rightarrow t' \Downarrow_\tau v$, and $d \lhd_\tau t$, then $d \lhd_\tau t'$.

$t \Downarrow_{\tau \to \tau'} v \Rightarrow t' \Downarrow_{\tau \to \tau'} v$ $\qquad u \in PCF_{\tau}$

$t\,u \Downarrow_{\tau'} v \Rightarrow t'\,u \Downarrow_{\tau'} v$

$$\Phi(\Gamma, t, \tau) := \forall \sigma \in \prod_{x \in dom(\Gamma)} PCF_{\Gamma(x)}, \forall \rho \in [\![\Gamma]\!], \sigma \triangleleft_\Gamma \rho \Rightarrow$$

$$[\![t]\!](e) \triangleleft_\tau t[\sigma]$$

$$\frac{}{\Gamma \vdash 0 : nat}$$

we need $[\![0]\!](e) \triangleleft_{nat} 0[\sigma]$

$$0 \in \mathbb{N}_\bot \triangleleft_{nat} 0$$

$$d \triangleleft_{nat} t := (d \in \mathbb{N} \Rightarrow t \Downarrow_{nat} d)$$

$$0 \Downarrow_{nat} 0$$

succ: $\dfrac{\Gamma \vdash t : nat}{\Gamma \vdash succ\ t : nat}$ take e, σ ...

IH: $[\![t]\!](e) \lhd_{nat} t[\sigma]$

we need $[\![succ\ t]\!](e) \lhd_{nat} (succ\ t)[\sigma]$

$$succ_\bot([\![t]\!](e)) \qquad succ(t[\sigma])$$

it is sufficient to show: $d \lhd_{nat} u \rightarrow succ_\bot(d) \lhd_{nat} succ(u)$

$succ_\bot(d) \in \mathbb{N} \Rightarrow succ(u) \Downarrow_{nat} \underline{succ_\bot(d)}$ $\qquad \dfrac{u \Downarrow_{nat} v}{succ(u) \Downarrow_{nat} succ(v)}$

$d \in \mathbb{N} \Rightarrow u \Downarrow_{nat} \underline{d}$ $\qquad = succ(\underline{d})$

fix: $\dfrac{\Gamma \vdash f : \tau \to \tau}{\Gamma \vdash \text{fix} f : \tau}$ $\quad e, \sigma$

IH: $[\![f]\!](e) \lhd_{\tau \to \tau} (f[\sigma])$

we need $[\![\text{fix} f]\!](e) \lhd_{\tau} (\text{fix} f)[\sigma]$

$$\text{fix}\,[\![f]\!](e) \lhd_{\tau} \text{fix}(f[\sigma])$$

enough: $d \lhd_{\tau \to \tau} g \Rightarrow \text{fix}\, d \lhd_{\tau} \text{fix}\, g$

Scott induction! $S = \{ e \in [\![\tau]\!] \,/\, e \lhd_{\tau} \text{fix}\, g \}$
contains $\perp$ and chain closed (1, 2 before)

5 stable under d: $e \in S$ ($e \lhd_\tau \text{fix } g$)

show $d(e) \in S$

assumption: $d \lhd_{\tau \to \tau} g \iff \forall e \in [\![\tau]\!], \forall u \in PCF_\tau, e \lhd_\tau u$,
$d(e) \lhd_\tau g(u)$

$d(e) \lhd_\tau \text{fix } g$ but $d(e) \lhd_\tau g(\text{fix } g)$

but by 3. $\dfrac{g(\text{fix } g) \Downarrow v}{\text{fix } g \Downarrow v}$ so $d(e) \lhd_\tau \text{fix } g$

Relating Denotational and Operational Semantics

Extensionality

Contextual preorder is the one-sided version of contextual equivalence: $\Gamma \vdash t \leq_{\mathrm{ctx}} t' : \tau$ if for all $\mathcal{C}$ such that $\cdot \vdash_{\Gamma,\tau} \mathcal{C} : \gamma$ and for all values v,

$$\mathcal{C}[t] \Downarrow_\gamma v \Rightarrow \mathcal{C}[t'] \Downarrow_\gamma v.$$

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$$\Gamma \vdash t \cong_{\text{ctx}} t' : \tau \Leftrightarrow (\Gamma \vdash t \leq_{\text{ctx}} t' : \tau \wedge \Gamma \vdash t' \leq_{\text{ctx}} t : \tau)$$

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It corresponds to formal approximation: for all PCF types τ and closed terms $t_1, t_2 \in \mathrm{PCF}_\tau$

$$t_1 \leq_{\mathrm{ctx}} t_2 : \tau \Leftrightarrow [\![t_1]\!] \lhd_\tau t_2.$$

Lemma: application contexts

For contextual preorder between closed terms, applicative contexts are enough.

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Let t_1, t_2 be closed terms of type τ. Then $t_1 \leq_{\mathrm{ctx}} t_2 : \tau$ if and only if, for every term $f : \tau \to \texttt{bool}$,

$$f\ t_1 \Downarrow_{\texttt{bool}} \texttt{true} \Rightarrow f\ t_2 \Downarrow_{\texttt{bool}} \texttt{true}.$$

For $\gamma = \texttt{bool}$ or $\texttt{nat}$, $t_1 \leq_{\text{ctx}} t_2 : \tau$ holds if and only if

$$\forall v.\, (t_1 \Downarrow_\gamma v \Rightarrow t_2 \Downarrow_\gamma v).$$

Extensionality properties of contextual preorder

For $\gamma = \texttt{bool}$ or $\texttt{nat}$, $t_1 \leq_{\mathrm{ctx}} t_2 : \tau$ holds if and only if

$$\forall v.\ (t_1 \Downarrow_\gamma v \Rightarrow t_2 \Downarrow_\gamma v).$$

At a function type $\tau \to \tau'$, $t_1 \leq_{\mathrm{ctx}} t_2 : \tau \to \tau'$ holds if and only if

$$\forall t \in \mathrm{PCF}_\tau\,.\ (t_1\ t \leq_{\mathrm{ctx}} t_2\ t : \tau').$$