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- We need to construct domains and continuous functions!

- Flat domains: "base" cases ($\mathbb{N}_\bot$, but also $\mathbb{B}_\bot$)
- Products of domains are domains, everything is componentwise

Let D_1 and D_2 be cpos. The **projections**

$$\pi_1 : \begin{array}{rcl} D_1 \times D_2 & \to & D_1 \\ (d_1, d_2) & \mapsto & d_1 \end{array} \qquad \pi_2 : \begin{array}{rcl} D_1 \times D_2 & \to & D_2 \\ (d_1, d_2) & \mapsto & d_2 \end{array}$$

are continuous functions.

$$\pi_1\left(\bigsqcup_n (d_{1,n}, d_{2,n})\right) = \pi_1\left(\bigsqcup_n d_{1,n}, \bigsqcup_n d_{2,n}\right)$$

$$= \bigsqcup_n d_{1,n} = \bigsqcup_n \pi_1(d_{1,n}, d_{2,n})$$

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are continuous functions.

If $f_1 : D \to D_1$ and $f_2 : D \to D_2$ are continuous functions from a cpo D, then the **pairing** function

$$\begin{array}{rrcl} \langle f_1, f_2 \rangle : & D & \to & D_1 \times D_2 \\ & d & \mapsto & (f_1(d), f_2(d)) \end{array}$$

is continuous.

The conditional function

$$\begin{aligned} \mathrm{if}: \quad \mathbb{B}_\perp \times (D \times D) &\to D \\ (x, d) &\mapsto \begin{cases} \pi_1(d) & \text{if } x = \mathrm{true} \\ \pi_2(d) & \text{if } x = \mathrm{false} \\ \perp_D & \text{if } x = \perp \end{cases} \end{aligned}$$

is continuous.

$$(b_n, d_{1,n}, d_{2,n}) \in (\mathbb{B}_\bot \times (D_1 \times D_2))_{n\in\mathbb{N}}$$

$$\bigsqcup_n (b_n, d_{1,n}, d_{2,n}) = (\bigsqcup_n b_n, \bigsqcup_n d_{1,n}, \bigsqcup_n d_{2,n})$$

$$if(\bigsqcup_n (b_n, d_{1,n}, d_{2,n})) \stackrel{?}{=} \bigsqcup_n if(b_n, d_{1,n}, d_{2,n})$$

it's enough to look at b_n constant $= b$

- $b = \bot$ then $if(\bot, d_{1,\infty}, d_{2,\infty}) = \bot = \bigsqcup_n \bot = \bigsqcup_n if(\bot, d_{1,n}, d_{2,n})$
- $b = true$ then $if(true, \bigsqcup_n d_{1,n}, \bigsqcup_n d_{2,n}) = \bigsqcup_n d_{1,n} = \bigsqcup_n if(true, d_{1,n}, d_{2,n})$
- $b = false$ similar

General product

Given a set I, suppose that for each $i \in I$ we are given a set X_i. The (cartesian) **product** of the X_i is

$$\prod_{i \in I} X_i$$

Two ways to see it:

- tuples: $(\dots, x_i, \dots)_{i \in I}$ such that $x_i \in X_i$;

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Projections (for any $i \in I$):

$$\pi_i : \left(\prod_{i \in I} X_i\right) \to X_i$$

General product of domains

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I-indexed products of cpos (domains) are cpos (domains), and projections are continuous.

Constructions on Domains

Function domains

Cpo/domain of continuous functions

Given two cpos $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$, the **function cpo** $(D \to E, \sqsubseteq)$ has underlying set

$$\{f : D \to E \mid \text{ is a } \textit{continuous} \text{ function}\}$$

equipped with the pointwise order:

$$f \sqsubseteq f' \overset{\text{def}}{\Leftrightarrow} \forall d \in D.\ f(d) \sqsubseteq_E f'(d).$$

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$$\frac{f \sqsubseteq_{D \to E} g \qquad x \sqsubseteq_D y}{f(x) \sqsubseteq_E g(y)}$$

Cpo/domain of continuous functions

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$$f \sqsubseteq f' \overset{\text{def}}{\Leftrightarrow} \forall d \in D.\ f(d) \sqsubseteq_E f'(d).$$

Argumentwise least elements and lubs:

$$\bot_{D \to E}(d) = \bot_E \qquad \left(\bigsqcup_{n \geq 0} f_n\right)(d) = \bigsqcup_{n \geq 0} f_n(d)$$

Take $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \dots \in D \to E$

Take $d \in D$ $\quad f_0(d) \sqsubseteq f_1(d) \sqsubseteq \dots$ $\quad f_\infty : d \mapsto \bigsqcup_n (f_n(d))$

So $\bigsqcup_n (f_n(d))$ exists it is an upper bound for $f_n(d)$

So $f_\infty \sqsupseteq f_n$ ie f_∞ is bound

Assume $f \sqsupseteq f_n$ then $f_n(d) \sqsubseteq f(d)$ for all d

so $\bigsqcup_n f_n(d) \sqsubseteq f(d)$

so $f_\infty(d) \sqsubseteq f(d)$

so $f_\infty \sqsubseteq f$

Remains: f_∞ is continuous

$d_0 \sqsubseteq d_1 \sqsubseteq d_m \dots$ chain in D

$$f_\infty\left(\bigsqcup_m d_m\right) = \bigsqcup_n \left(f_n\left(\bigsqcup_m d_m\right)\right) = \bigsqcup_n \bigsqcup_m f_n(d_m)$$

$$= \bigsqcup_m \bigsqcup_n f_n(d_m)$$

$$= \bigsqcup_m f_\infty(d_m)$$

Function operations are continuous

Evaluation, currying ($f : (D' \times D) \to E$) and composition

$$\begin{array}{rccc} \mathrm{eval} : & (D \to E) \times D & \to & E \\ & (f, d) & \mapsto & f(d) \end{array}$$

$$\begin{array}{rccc} \mathrm{cur}(f) : & D' & \to & (D \to E) \\ & d' & \mapsto & \lambda d \in D.\, f(d', d) \end{array}$$

$$\begin{array}{rccc} \circ : & \big((E \to F) \times (D \to E)\big) & \longrightarrow & (D \to F) \\ & (f, g) & \mapsto & \lambda d \in D.\, g(f(d)) \end{array}$$

are all well-defined and continuous.

$f : (D \times D') \to E$ then $\mathrm{cur}(f) : D \to (D' \to E)$ is continuous

$\mathrm{cur} : ((D \times D') \to E) \to (D \to (D' \to E))$ is continuous?

$eval : D \times (D \to E) \to E$

Take $d_0 \sqsubseteq d_1 \sqsubseteq \ldots d_n \in D$ $\quad f_0 \sqsubseteq f_1 \sqsubseteq \ldots \in D \to E$

$$eval\left(\bigsqcup_n (f_n, d_n)\right) = eval\left(\bigsqcup_n f_n, \bigsqcup_n d_n\right) = eval\left(\bigsqcup_m f_m, \bigsqcup_n d_n\right)$$

$$= \left(\bigsqcup_m f_m\right)\left(\bigsqcup_n d_n\right)$$

$$= \bigsqcup_m \left(f_m\left(\bigsqcup_n d_n\right)\right)$$

$$= \bigsqcup_m \bigsqcup_n f_m(d_n)$$

$$= \bigsqcup_k f_k(d_k)$$

$$= \bigsqcup_k eval(f_k, d_k)$$

Continuity of the fixed point operator

$$\text{fix}: \quad (D \to D) \quad \to \quad D$$

is continuous.

Take $f_0 \sqsubseteq f_1 \sqsubseteq \dots f_n \in D \to D$

First show fix monotone : $f \sqsubseteq e \in D \to D$

$f(\mathrm{fix}(e)) \sqsubseteq e(\mathrm{fix}(e))$ bc $f \sqsubseteq e$

$\sqsubseteq \mathrm{fix}(e)$ so $\mathrm{fix}(e)$ is a prefixed point of f

$\mathrm{fix}(f) \sqsubseteq \mathrm{fix}(e)$

We want $\mathrm{fix}(\bigsqcup_n f_n) = \bigsqcup_n \mathrm{fix}(f_n)$ it is enough to show

$\mathrm{fix}(\bigsqcup_n f_n) \sqsubseteq \bigsqcup_n \mathrm{fix}(f_n)$

enough: $\bigsqcup_n \mathrm{fix}(f_n)$ is a pre-fixed point for $\bigsqcup_n f_n$

$$\left(\bigsqcup_m f_m\right)\left(\bigsqcup_n fix(f_n)\right) = \bigsqcup_m \bigsqcup_n f_m(fix(f_n))$$

$$= \bigsqcup_k f_k(fix(f_k))$$

$$= \bigsqcup_k fix(f_k)$$

So $\bigsqcup_k fix(f_k)$ is a fixed point of $\bigsqcup_m f_m$

Constructions on Domains

Back to the introduction

$$[\![\texttt{while}\ X > 0\ \texttt{do}\ (Y := X * Y; X := X - 1)]\!]$$

is a fixed point of the following $F : D \to D$, where D is $(\mathbf{State} \rightharpoonup \mathbf{State})$:

$$F(w)([X \mapsto x, Y \mapsto y]) = \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ w([X \mapsto x - 1, Y \mapsto x \cdot y]) & \text{if } x > 0. \end{cases}$$

$$[\![\texttt{while}\ X > 0\ \texttt{do}\ (Y := X * Y; X := X - 1)]\!]$$

is a fixed point of the following $F : D \to D$, where D is $(\text{State}_\perp \to \text{State}_\perp)$:

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$$F(\perp) = \perp$$

$\text{State}_\perp \to \text{State}_\perp$ is a domain!

Kleene's fixed point theorem

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$$w_\infty = \bigsqcup_{i \in \mathbb{N}} F^n(\bot)$$

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We **can** compute explicitly

$$w_\infty[X \mapsto x, Y \mapsto y] = \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x < 0 \\ [X \mapsto 0, Y \mapsto (x!) \cdot y] & \text{if } x \geq 0 \end{cases}$$

And **check** this agrees with the operational semantics.