Where we're at

- The (denotational) semantic of recursive definitions (*e.g.* loops) is a fixed point: some w such that $F(w) = w$;
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- The (denotational) semantic of recursive definitions (*e.g.* loops) is a fixed point: some w such that $F(w) = w$;
- it seems we can compute such fixed points by iterating F on some minimal element.

- If D is a poset and $f : D \to D$ is monotone and has a least pre-fixed point $\mathrm{fix}\, f$ then $f(\mathrm{fix}(f)) = \mathrm{fix}(f)$.

Least Fixed Points

Least upper bounds

Least upper bound of a chain

The **least upper bound** of countable increasing chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$, written $\bigsqcup_{n \geq 0} d_n$, satisfies the two following properties:

$$\text{LUB-BOUND}\ \frac{}{x_i \sqsubseteq \bigsqcup_{n \geq 0} x_n} \qquad \text{LUB-LEAST}\ \frac{\forall n \geq 0 \,.\, x_n \sqsubseteq x}{\bigsqcup_{n \geq 0} x_n \sqsubseteq x}$$

Lubs are unique.

Bound: $x_n \le x_n$

$\forall n.\; x_n \le \bigsqcup'_n x_n$

Least: $\bigsqcup_n x_n \le \bigsqcup'_n x_n$ $\quad$ $\bigsqcup'_n x_n \le \bigsqcup_n x_n$

Asym: $\bigsqcup_n x_n = \bigsqcup'_n x_n$

Properties of lubs

Lubs are unique.

Lubs are monotone: if for all $n \in \mathbb{N}.\ d_n \sqsubseteq e_n$, then $\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n$.

$$\text{Least}\ \dfrac{\forall n.\ \ \text{Bound}\ \dfrac{d_n \leq e_n}{d_n \leq \bigsqcup_n e_n}}{\bigsqcup_n d_n \leq \bigsqcup_n e_n}$$

Properties of lubs

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$$\textsc{lub-mon}\ \frac{\forall i.\ d_i \sqsubseteq e_i}{\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n}$$

Properties of lubs

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Lubs are monotone: if for all $n \in \mathbb{N}.\ d_n \sqsubseteq e_n$, then $\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n$.

For any d, $\bigsqcup_n d = d$.

Lubs are unique.

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For any d, $\bigsqcup_n d = d$.

$d_0 \sqsubseteq d_1 \cdots \sqsubseteq d_N \sqsubseteq d_{N+1}$

For any chain and $N \in \mathbb{N}$, $\bigsqcup_n d_n = \bigsqcup_n d_{n+N}$.

Properties of lubs

Lubs are unique (if they exist).

Lubs are monotone: if for all $n \in \mathbb{N}.\ d_n \sqsubseteq e_n$, then $\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n$ (if they exist).

For any d, $\bigsqcup_n d = d$ (and in particular it exists).

For any chain and $N \in \mathbb{N}$, $\bigsqcup_n d_n = \bigsqcup_n d_{n+N}$ (if any of the two exists).

Diagonalisation

Assume $d_{m,n} \in D$ $(m, n \geq 0)$ satisfies

$$m \leq m' \wedge n \leq n' \Rightarrow d_{m,n} \sqsubseteq d_{m',n'}.$$

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Then, assuming they exist, the lubs form two chains

$$\bigsqcup_{n \geq 0} d_{0,n} \sqsubseteq \bigsqcup_{n \geq 0} d_{1,n} \sqsubseteq \bigsqcup_{n \geq 0} d_{2,n} \sqsubseteq \ldots$$

and

$$\bigsqcup_{m \geq 0} d_{m,0} \sqsubseteq \bigsqcup_{m \geq 0} d_{m,1} \sqsubseteq \bigsqcup_{m \geq 0} d_{m,3} \sqsubseteq \ldots$$

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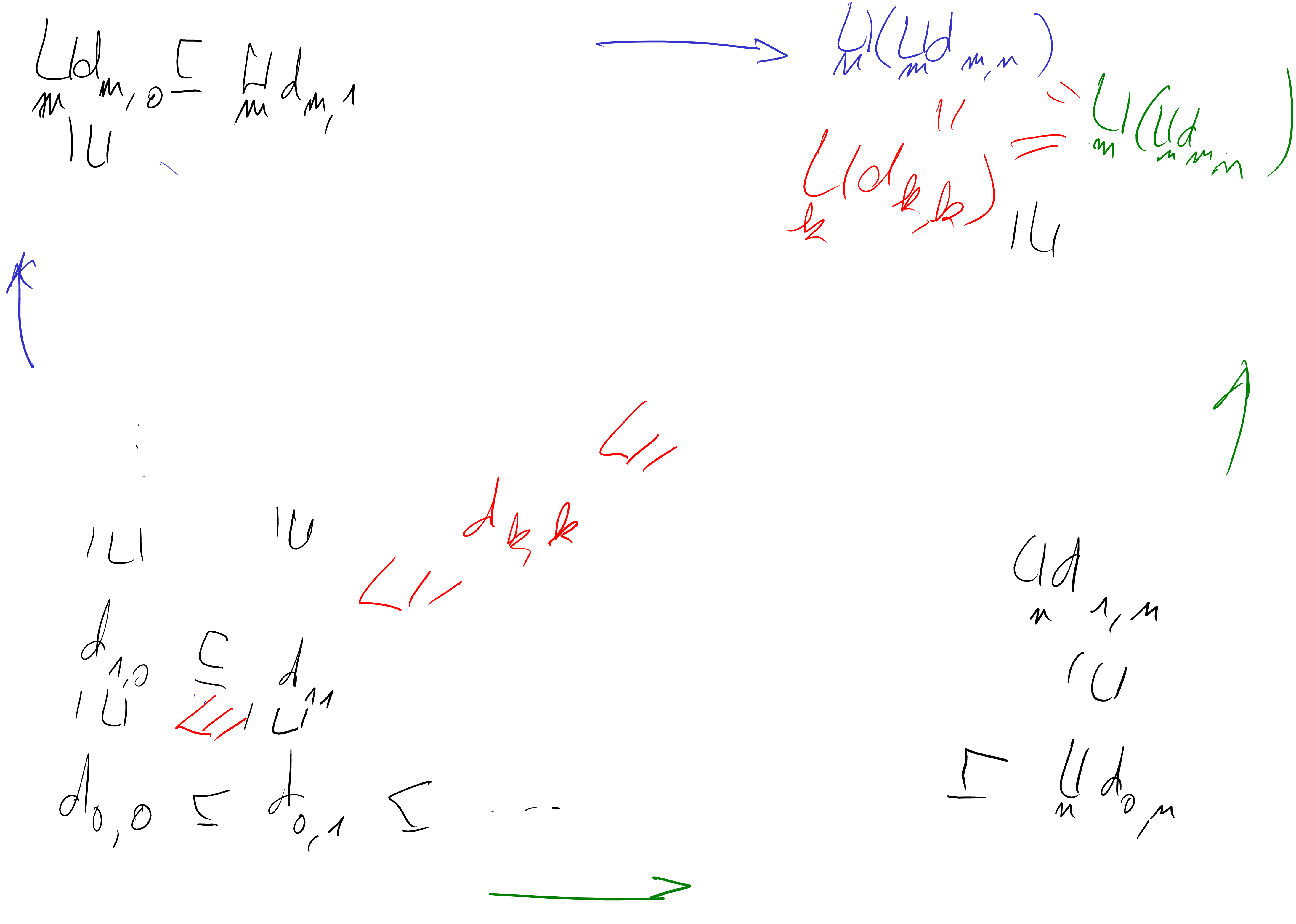
Moreover, again assuming they exist,

$$\bigsqcup_{m\geq 0}\left(\bigsqcup_{n\geq 0} d_{m,n}\right) = \bigsqcup_{k\geq 0} d_{k,k} = \bigsqcup_{n\geq 0}\left(\bigsqcup_{m\geq 0} d_{m,n}\right) .$$



$\bigsqcup_m d_{m,0} \sqsubseteq \bigsqcup_m d_{m,1}$
$\bigsqcup_m \left(\bigsqcup_n d_{m,n}\right)$
$= \bigsqcup_n \left(\bigsqcup_m d_{m,n}\right)$
$\bigsqcup_k d_{k,k}$
$d_{k,k}$
$d_{1,0} \sqsubseteq d_{1,1}$
$d_{0,0} \sqsubseteq d_{0,1} \sqsubseteq \ldots$
$\bigsqcup_n d_{1,n}$
$\sqsubseteq \bigsqcup_n d_{0,n}$

$$\dfrac{}{}$$

Bound $\dfrac{}{d_{m,n} \sqsubseteq d_{max(m,n)}}$ ...

Least $\dfrac{\forall n.\ d_{m,n} \sqsubseteq \bigsqcup_k d_{k,k}}{}$

Least $\dfrac{\forall m.\ \bigsqcup_n d_{m,n} \sqsubseteq \bigsqcup_k d_{k,k}}{\bigsqcup_m \bigsqcup_n d_{m,n} \sqsubseteq \bigsqcup_k d_{k,k}}$

Bound $\dfrac{}{d_{k,k} \sqsubseteq d_{k,k}}$

Bound $\dfrac{d_{k,k} \sqsubseteq \bigsqcup_n d_{k,n}}{}$

Least $\dfrac{\forall k.\ d_{k,k} \sqsubseteq \bigsqcup_m \bigsqcup_n (d_{m,n})}{\bigsqcup_k d_{k,k} \sqsubseteq \bigsqcup_m (\bigsqcup_n d_{m,n})}$

Asym $\dfrac{}{\bigsqcup_m (\bigsqcup_n d_{m,n}) = \bigsqcup_k d_{k,k}}$

Least Fixed Points

Complete partial orders and domains

Cpos and domains

A chain complete poset/cpo is a poset $(D, \sqsubseteq)$ in which all chains have least upper bounds.

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Beware: the lub need only exist if the x_i form a chain!

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Beware: the lub need only exist if the x_i form a chain!

A domain is a cpo with a least element $\perp$.

Least element: $\bot$ is the totally undefined function.

Domain of partial functions

Least element: $\bot$ is the totally undefined function.

Lub of a chain: $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \ldots$ has lub f such that

$$f(x) = \begin{cases} f_n(x) & \text{if } x \in \mathrm{dom}(f_n) \text{ for some } n \\ \text{undefined} & \text{otherwise} \end{cases}$$

Domain of partial functions

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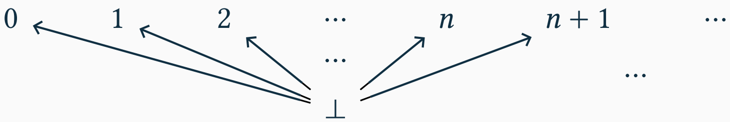
Lub of a chain: $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \ldots$ has lub f such that

$$f(x) = \begin{cases} f_n(x) & \text{if } x \in \mathrm{dom}(f_n) \text{ for some } n \\ \text{undefined} & \text{otherwise} \end{cases}$$

Beware: the definition of $\bigsqcup_{n \geq 0} f_n$ is unambiguous only if the f_i form a chain!

The flat natural numbers $\mathbb{N}_\perp$



Least Fixed Points

Continuous functions

Given two cpos D and E, a function $f\colon D \to E$ is continuous if

- it is monotone, and
- it preserves lubs of chains, *i.e.* for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in D, we have

$$f(\bigsqcup_{n \geq 0} d_n) = \bigsqcup_{n \geq 0} f(d_n)$$

Given two cpos D and E, a function $f\colon D \to E$ is **continuous** if

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$$f(\bigsqcup_{n\geq 0} d_n) = \bigsqcup_{n\geq 0} f(d_n)$$

Bound: $d_n \sqsubseteq \bigsqcup_n d_n$

Mon: $\forall n\ f(d_n) \sqsubseteq f(\bigsqcup_n d_n)$

Least: $\bigsqcup_n f(d_n) \sqsubseteq f(\bigsqcup_n d_n)$

A function f is **strict** if $f(\bot_D) = \bot_E$.

All computable functions are continuous.

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The typical non-continuous function: "is a sequence the constant $\mathbf{0}$"?

$$0 \quad 0 \quad \bot \quad \dots \qquad \mapsto \bot$$
$$0 \quad 0 \quad 0 \quad 0 \quad 1 \quad \dots \qquad \mapsto 1$$

$$0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \overline{0} \qquad \mapsto 0$$

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The typical non-continuous function: “is a sequence the constant $\mathbf{0}$”?

$$
\begin{array}{llllll l}
0 & 0 & \bot & \ldots & & & \mapsto \bot \\
0 & 0 & 0 & 0 & 1 & \ldots & \mapsto 1 \\
0 & 0 & 0 & 0 & 0 & \ldots & \mapsto ? \\
\\
0 & 0 & 0 & 0 & 0 & \overline{0} & \mapsto 0
\end{array}
$$

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Intuition: non-continuity ≈ "jump at infinity" ≈ non-computability

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Intuition: non-continuity ≈ “jump at infinity” ≈ non-computability

Later in the course: show the thesis... by giving a denotational semantics.

Least Fixed Points

Kleene's fixed point theorem

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Let $f \colon D \to D$ be a continuous function on a domain D. Then f possesses a least pre-fixed point, given by

$$\mathrm{fix}(f) = \bigsqcup_{n \geq 0} f^n(\bot).$$

$\bot \sqsubseteq f(\bot)$

$f(\bot) \sqsubseteq f^2(\bot) \sqsubseteq f^3(\bot) \ldots$

Kleene's fixed point theorem

Let $f\colon D \to D$ be a continuous function on a domain D. Then f possesses a least pre-fixed point, given by

$$\mathrm{fix}(f) = \bigsqcup_{n \geq 0} f^n(\perp).$$

It is thus also the least fixed point of f!

Take e a fixed point of f

$$\frac{\forall n\; f^n(\bot) \sqsubseteq e}{\bigsqcup_n f^n(\bot) \sqsubseteq e}$$

By induction on n:

$f^0(\bot) = \bot \sqsubseteq e$

$f^{n+1}(\bot) = f(f^n(\bot)) \overset{IH}{\sqsubseteq} f(e) \sqsubseteq e$

So $\bigsqcup_n f^n(\bot)$ is smaller than all the fixed points

$$f\left(\bigsqcup_n f^n(\bot)\right) \sqsubseteq \bigsqcup_n f^n(\bot)$$

monotonicity $\overset{\text{\tiny U}}{\bigsqcup_n} f^{n+1}(\bot)$ "

Constructions on Domains

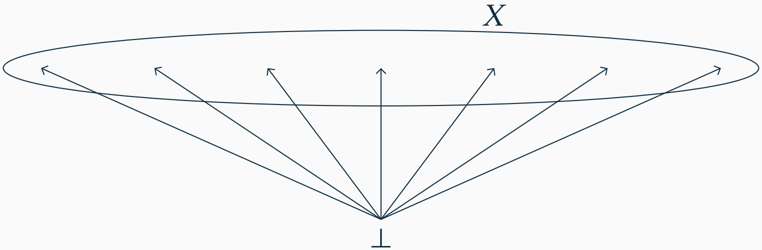
Constructions on Domains

Flat domains

Flat domain on X

The flat domain on a set X is defined by:

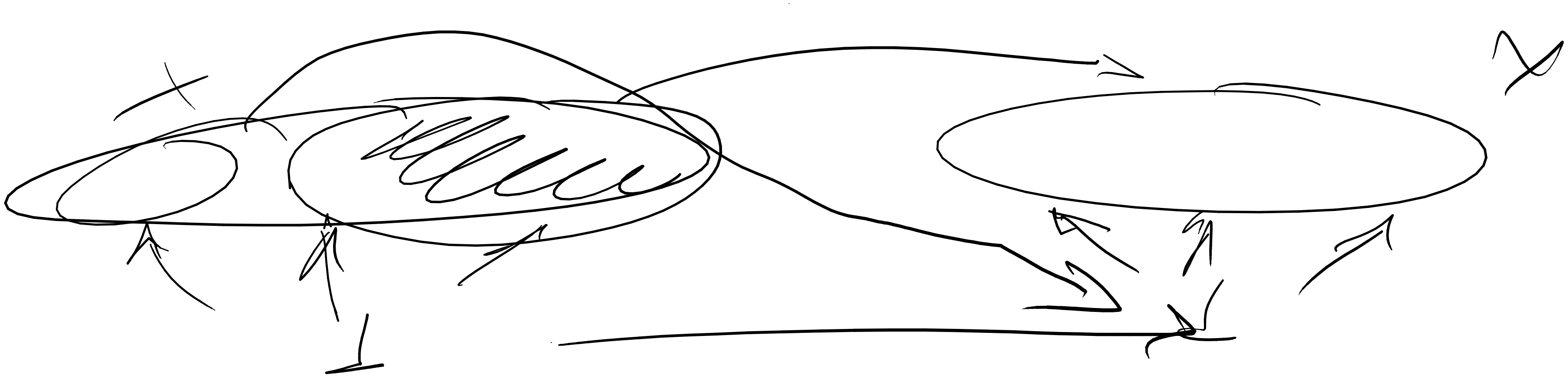
- its underlying set $X \uplus \{\bot\}$;
- $x \sqsubseteq x'$ if either $x = \bot$ or $x = x'$.



Let $f : X \rightharpoonup Y$ be a partial function between two sets. Then

$$\begin{aligned} f_\perp : \quad X_\perp &\to Y_\perp \\ d &\mapsto \begin{cases} f(d) & \text{if } d \in X \text{ and } f \text{ is defined at } d \\ \perp & \text{if } d \in X \text{ and } f \text{ is not defined at } d \\ \perp & \text{if } d = \perp \end{cases} \end{aligned}$$

defines a continuous function between the corresponding flat domains.



Constructions on Domains

Products of domains

Binary product

The product of two posets $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ has underlying set

$$D_1 \times D_2 = \{(d_1, d_2) \mid d_1 \in D_1 \wedge d_2 \in D_2\}$$

and partial order $\sqsubseteq$ defined by

$$(d_1, d_2) \sqsubseteq (d_1', d_2') \overset{\text{def}}{\Leftrightarrow} d_1 \sqsubseteq_1 d_1' \wedge d_2 \sqsubseteq_2 d_2'$$

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$$\text{POX}\ \frac{d_1 \sqsubseteq_1 d_1' \qquad d_2 \sqsubseteq_2 d_2'}{(d_1, d_2) \sqsubseteq (d_1', d_2')}$$

Componentwise lubs and least elements

lubs of chains are computed componentwise:

$$\bigsqcup_{n\geq 0}(d_{1,n}, d_{2,n}) = (\bigsqcup_{i\geq 0} d_{1,i}, \bigsqcup_{j\geq 0} d_{2,j}).$$

$$(d_{1,n}, d_{2,n}) \sqsubseteq (d_{1,n+1}, d_{2,n+1})$$

$$d_{1,n} \sqsubseteq d_{1,n+1} \wedge d_{2,n} \sqsubseteq d_{2,n+1}$$

Componentwise lubs and least elements

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$$\bigsqcup_{n \geq 0} (d_{1,n}, d_{2,n}) = (\bigsqcup_{i \geq 0} d_{1,i}, \bigsqcup_{j \geq 0} d_{2,j}).$$

If $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ have least elements, so does $(D_1 \times D_2, \sqsubseteq)$ with

$$\bot_{D_1 \times D_2} = (\bot_{D_1}, \bot_{D_2})$$

$(\bigsqcup_n d_{1,n}, \bigsqcup_n d_{2,n})$ is a lub

Say (e_1, e_2) is an ub $(d_{1,n}, d_{2,n}) \sqsubseteq (e_1, e_2)$

$d_{1,n} \sqsubseteq e_1 \Rightarrow \bigsqcup_n d_{1,n} \sqsubseteq e_1$

$d_{2,n} \sqsubseteq e_2 \Rightarrow \bigsqcup_n d_{2,n} \sqsubseteq e_2$

$\Big\} (\bigsqcup_n d_{1,n}, \bigsqcup_n d_{2,n}) \sqsubseteq (e_1, e_2)$

$(d_{1,n}, d_{2,n}) \sqsubseteq (\bigsqcup_n d_{1,n}, \bigsqcup_n d_{2,n})$

$\Uparrow$

$d_{1,n} \sqsubseteq \bigsqcup_n d_{1,n}$

$d_{2,n} \sqsubseteq \bigsqcup_n d_{2,n}$

Componentwise lubs and least elements

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$$\bigsqcup_{n\geq 0}(d_{1,n}, d_{2,n}) = (\bigsqcup_{i\geq 0} d_{1,i}, \bigsqcup_{j\geq 0} d_{2,j}).$$

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$$\bot_{D_1\times D_2} = (\bot_{D_1}, \bot_{D_2})$$

Products of cpos (domains) are cpos (domains).

Functions of two arguments

A function $f : (D \times E) \to F$ is monotone if and only if it is monotone in each argument separately:

$$\forall d, d' \in D, e \in E.\, d \sqsubseteq d' \Rightarrow f(d, e) \sqsubseteq f(d', e)$$
$$\forall d \in D, e, e' \in E.\, e \sqsubseteq e' \Rightarrow f(d, e) \sqsubseteq f(d, e').$$

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$$\forall d \in D, e, e' \in E.\, e \sqsubseteq e' \Rightarrow f(d,e) \sqsubseteq f(d,e').$$

Moreover, it is continuous if and only if it preserves lubs in each argument separately:

$$f(\bigsqcup_{m\geq 0} d_m\,,\; e) = \bigsqcup_{m\geq 0} f(d_m, e)$$
$$f(d\,,\; \bigsqcup_{n\geq 0} e_n) = \bigsqcup_{n\geq 0} f(d, e_n).$$

Derived rules for functions of two arguments

$$\text{MON}\times\ \frac{f \text{ monotone} \qquad x \sqsubseteq x' \qquad y \sqsubseteq y'}{f(x, y) \sqsubseteq f(x', y')}$$

$$f\left(\bigsqcup_m x_m, \bigsqcup_n y_n\right) = \bigsqcup_m \bigsqcup_n f(x_m, y_n) = \bigsqcup_k f(x_k, y_k)$$