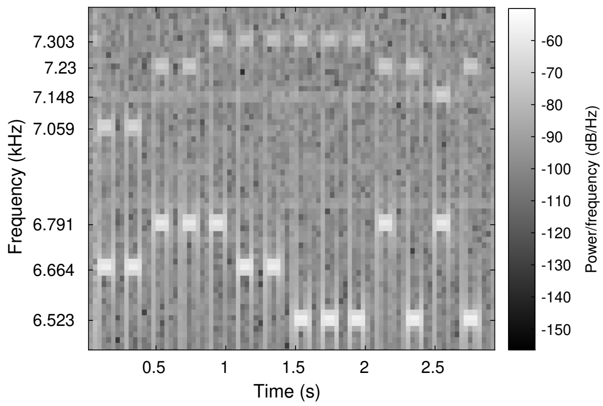
11 Digital Signal Processing (mgk25)

This question can only be attempted by Part II 50% candidates.

You are the new CTO of *Missampled Ltd*, a consulting company specializing in fixing digital-signal-recording accidents. These are the first customers seeking your help:

(*a*) A police officer has recorded a conversation between suspects on an analog phone line. The recording $\{x_n\}$ has sampling frequency $f_s = 16$ kHz. But the officer had accidentally activated a "scramble" switch on the recorder, and as a result the recording now sounds high-pitched and is unintelligible.



The manual of the recorder does not explain what the "scramble" switch does. Using a spectrogram (above), you spot at the start of the recording a sequence of 14 tone pairs.

(*i*) What six computational steps are typically involved in producing such a spectrogram from a sequence of real-valued samples? [6 marks]

(*ii*) The spectrogram reminds you of DTMF-encoded touch-tone digits, but the frequencies are clearly not the standard ones at 697, 770, 852, 941, 1209, 1336, and 1477 Hz. What appears to have happened to the frequencies in this recording, how can this transformation be explained as a simple time-domain operation on its samples, and how can you then restore it such that the officer can hear the original voices again? [6 marks]

(*b*) A TV producer discovered that during the recording of a stage production, one of the microphones accidentally had activated the following digital FIR filter (where $\{x_n\}$ is the desired audio signal and $\{y_n\}$ is the available recorded sequence):

$$y_n = 0.4 \times \left(x_n + x_{n-1} + \frac{1}{2} x_{n-2} \right)$$

(*i*) What is the z-transform $H(z)$ of the impulse response of this filter? [2 marks]

(*ii*) What is the z-transform of the impulse response of a filter G that, if applied to the recorded samples $\{y_n\}$, converts them back into the original waveform $\{x_n\}$? [2 marks]

(*iii*) Draw a Direct Form I representation of G. [4 marks]

11 Digital Signal Processing (mgk25)

This question can only be attempted by Part II 50% candidates.

(*a*) Name one advantage and one disadvantage of Finite-Impulse-Response (FIR) filters over Infinite-Impulse-Response (IIR) filters. [2 marks]

(*b*) For each of the following discrete systems $\{y_n\} = T\{x_n\}$, either show that T is equivalent to a convolution operation, by providing an impulse response $\{h_n\}$ such that

$$y_n = \sum_{i=-\infty}^{\infty} h_i x_{n-i}$$

or explain why the system cannot be described through convolution.

(*i*) $y_n = \frac{1}{2}(x_{2n} + x_{2n+1})$ [2 marks]

(*ii*) $y_n = x_{n+4}$ [2 marks]

(*iii*) $y_n = \frac{3}{2}x_{n-1} - \frac{1}{2}y_{n-2}$ [4 marks]

(*c*) What is the z-transform of the impulse response of the system in Part (*b*)(*iii*)? [4 marks]

(*d*) Consider a digital filter where the z-transform of the impulse response is

$$H(z) = \frac{z^2 - 1}{z^2 + \frac{49}{64}}.$$

(*i*) Draw the location of poles and zeros of $H(z)$ in the z-plane. [2 marks]

(*ii*) What is this kind of filter called? [1 mark]

(*iii*) A test signal $x(t) = \cos(2\pi f t)$ is sampled into $x_n = x(n/f_\mathrm{s})$, with rate $f_\mathrm{s} = 4$ kHz, and then passed through this filter. For what values of f will the root-mean-square level at the filter output be maximal? [3 marks]

5 Digital Signal Processing (MGK)

Your friend Sam works on a physics experiment. This generates a voltage waveform $v(t)$ that is the sum of several signals:

- a sine wave $s(t) = A \cdot \sin(2\pi t f + \phi)$, the frequency f and phase ϕ of which are not known in advance, but f will be within 9.6 kHz $< f <$ 12.0 kHz;
- several other sine waves with frequencies below 8 kHz that Sam needs to ignore in her measurements;
- low levels of noise at all frequencies.

Sam needs to estimate the amplitude A of $s(t)$. She uses a USB audio recorder with a built-in 16 kHz anti-aliasing low-pass filter to digitize $v(t)$ at sampling frequency f_s = 48 kHz, recording s = 100 000 consecutive samples, resulting in real-valued samples $v_0, \ldots, v_{s-1}$. She implemented this algorithm to estimate A:

input $v_0, \ldots, v_{s-1}$

$b := 1000; \quad c := \left\lfloor \frac{s}{b} \right\rfloor$

$w_{k,l} := v_{kb+l}$ for all $0 \le k < c$, $0 \le l < b$

$$x_{k,n} := \sum_{m=0}^{b-1} w_{k,m} \cdot \mathrm{e}^{-2\pi \mathrm{j} \frac{nm}{b}} \qquad \text{for all } 0 \le k < c,\ 0 \le n < b$$

$$y_n := \left| \frac{1}{c} \cdot \sum_{k=0}^{c-1} x_{k,n} \right| \qquad \text{for all } 0 \le n < b$$

$z := \max\{y_{n_1}, \ldots, y_{n_2}\}$ with $n_1 = 200$, $n_2 = 220$

output z

(*a*) Sam hopes that $A \approx z \cdot \alpha$ for some calibration constant α. She tries to determine α by connecting the USB audio recorder's input to a calibrated laboratory sine-wave generator set to output an amplitude of "60.0 dBµV". What amplitude A in volts will this test signal $A \cdot \sin(\ldots)$ have? [3 marks]

(*b*) When Sam varies the test-signal frequency f in the range 9.6–12.0 kHz, she is disappointed that the output z varies greatly: for some f it even drops to zero!

Describe what Sam's algorithm tries to do, identify and explain *three* problems in it, and change *three* lines to make z more proportional to A across the expected range of f, and close to zero outside that range. [15 marks]

(*c*) Suggest a small adjustment to b to accommodate a faster algorithm for one of the above steps. [2 marks]

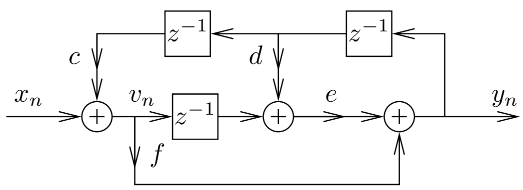
6 Digital Signal Processing (MGK)

(*a*) When converting a digital audio signal from one sampling frequency to another, it is common practice to use a low-pass filter. What is the purpose of this low-pass filter, and what cut-off frequency should it have if the change of sampling frequency is

(*i*) from 12 kHz to 48 kHz;

(*ii*) from 48 kHz to 12 kHz. [4 marks]

(*b*) You are working on the firmware of a quadcopter drone. Your colleague, through trial and error, found that the following recursive filter nicely avoids unwanted oscillations in the control system:



(*i*) What are the first three samples h_0, h_1, h_2 of the impulse response of this filter? [*Note:* All delay elements have been initialized to zero.] [6 marks]

(*ii*) What is the z-transform $H(z) = Y(z)/X(z)$ of the impulse response of this digital filter? [5 marks]

(*iii*) The software development kit of your flight controller can only implement digital filters of the form

$$y_n = \sum_{k=0}^{3} b_k \cdot x_{n-k} - \sum_{l=1}^{3} a_l \cdot y_{n-l}.$$

What coefficient values a_l and b_k ($0 \le k \le 3$, $1 \le l \le 3$) will implement the same impulse response as your colleague's filter? [5 marks]

6 Digital Signal Processing (MGK)

A zoologist wants to record the echo-location sounds emitted by a bat. The species of bat to be recorded emits only sounds in the frequency range 40 kHz to 80 kHz and the microphone used includes an analog filter with that passband.

(*a*) Explain for each of the following sampling techniques how it can be used to convert a continuous ultrasonic microphone signal $x(t)$ into a discrete-time sequence $\{x_n\}$ and state for each technique the lowest sampling frequency f_s that enables the exact reconstruction of $x(t)$ from $\{x_n\}$:

(*i*) Passband sampling [3 marks]

(*ii*) IQ downconversion [5 marks]

(*b*) Using a 32-bit floating-point data type, how many bytes per second are required to store each of the two resulting discrete sequences from part (*a*)? [2 marks]

(*c*) Compare your answers to part (*b*) with the memory required for storing $x(t)$ sampled at the Nyquist rate of 160 kHz and explain the difference in terms of redundancy in the acquired spectrum. [2 marks]

(*d*) If the sampling techniques from part (*a*) are applied to a test signal $x(t) = \cos(2\pi f t)$ with $f = 45$ kHz, what does the discrete-time Fourier transform of the resulting discrete sequence $\{x_n\}$ look like (over the normalized frequency range $-\pi < \dot{\omega} \leq \pi$) for each technique? [4 marks]

(*e*) For both sampling techniques described in part (*a*), briefly outline the steps needed to reconstruct the original continuous waveform from the discrete sequence. [4 marks]

6 Digital Signal Processing (MGK)

(*a*) A discrete-time LTI filter can be described through the locations of zeros and poles in the z-transform $H(z)$ of its impulse response. Consider an IIR filter of order 2 with $H(c_1) = H(c_2) = 0$ and $|H(z)| \to \infty$ for $z \to d_1$ and $z \to d_2$.

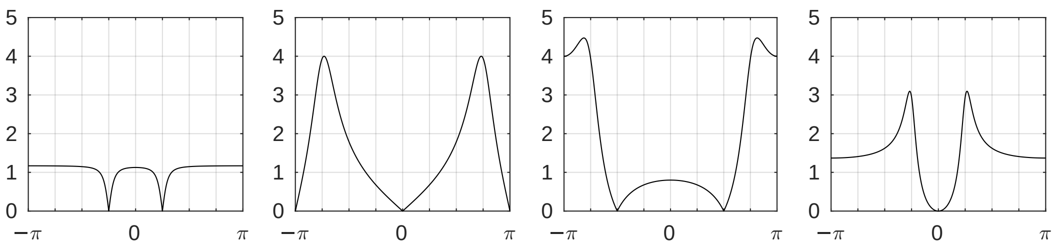
(*i*) What is the z-transform $H(z)$ of its impulse response? [2 marks]

(*ii*) What additional parameter (beyond c_1, c_2, d_1, d_2) is required to fully describe the impulse response of this filter? [1 mark]

(*iii*) What is the magnitude of the discrete-time Fourier transform (DTFT) of the impulse response of this filter? [2 marks]

(*iv*) Under what condition on c_1, c_2, d_1, and d_2 is the impulse-response of this filter real-valued? [2 marks]

(*b*) The following plots show the magnitude of the DTFT of the real-valued impulse response of four different IIR filters:

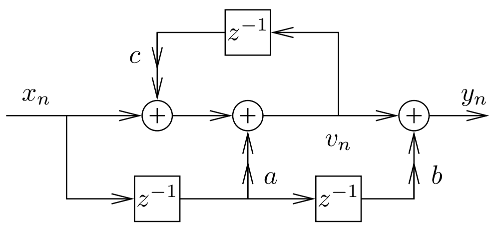


(*i*) (*ii*) (*iii*) (*iv*)

The z-transform of each impulse response has two zeros and two poles. Each zero or pole is at one of these 12 possible locations: $e^{\pi jk/4}$ with $k \in \{0, \ldots, 7\}$ or $0.6 \pm 0.6j$ or $-0.5 \pm 0.5j$.

For each filter, state the location of both zeros and both poles. Explain the reasoning behind your choice. [8 marks]

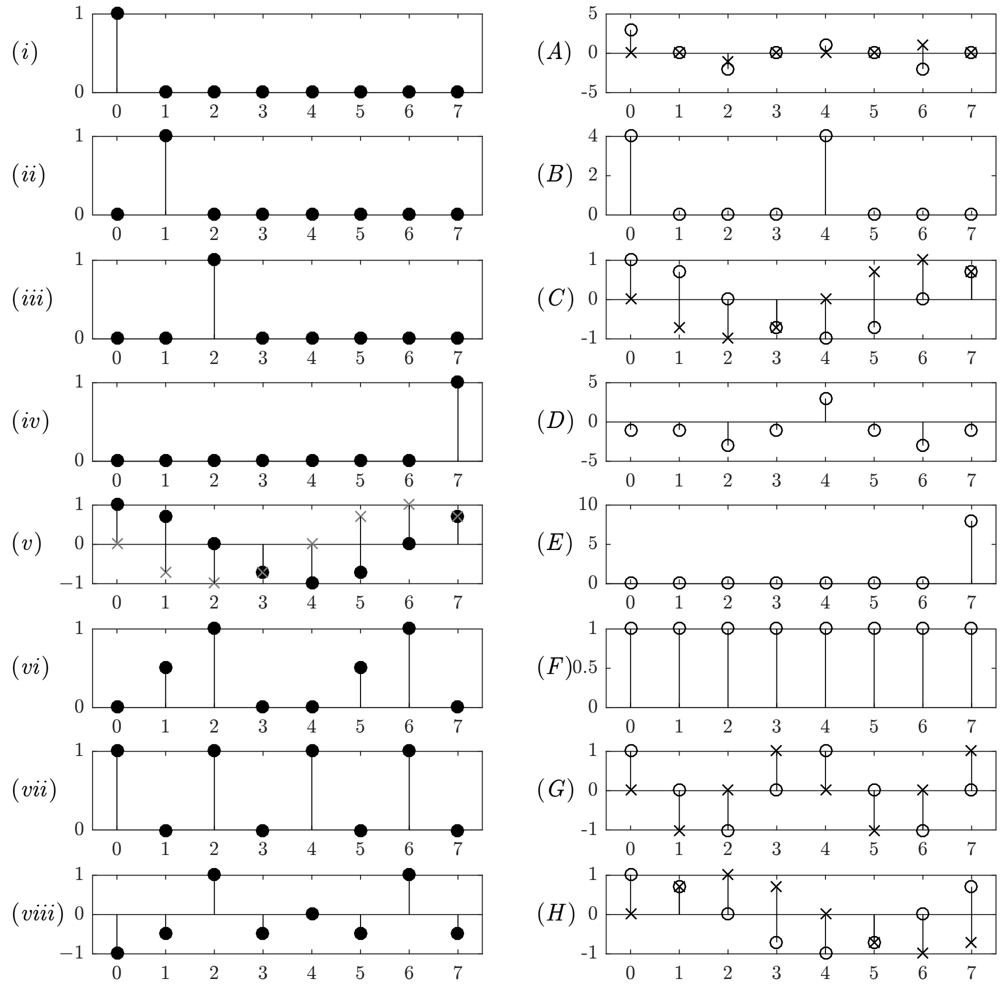
(*c*) What is the z-transform $H(z)$ of the impulse response of the following filter? [5 marks]



6 Digital Signal Processing (MGK)

(a) Figures (i)–($viii$) show eight different input vectors $x \in \mathbb{C}^8$. For each, identify one of figures (A)–(H) that shows the DFT output $X \in \mathbb{C}^8$ with $X_k = \sum_{n=0}^{7} x_n \cdot \mathrm{e}^{-2\pi \mathrm{j} kn/8}$.

Briefly explain each choice. Real components are shown as circles. For non-real vectors, the imaginary components are shown in addition as crosses. [8 marks]



(b) Are these statements true or false? Explain your answers. [3 marks each]

(i) The system $y_n = x_n + y_{n-1}$ has an impulse response with z-transform $\frac{1}{1+z}$.

(ii) A continuous signal can *only* be reconstructed after sampling if the sampling frequency is larger than twice the highest frequency in the signal.

(iii) Convolution of a signal with a triangular window function causes its power spectrum to be multiplied with a sinc^3 function.

(iv) To convert the z-transform $H(z)$ of the impulse response of any LTI filter into the z-transform of its step response, divide $H(z)$ by $1 - z^{-1}$.

6 Digital Signal Processing (MGK)

(*a*) Let δ be the Dirac delta function and $T, b > 0$ be time intervals. Give the Fourier transform

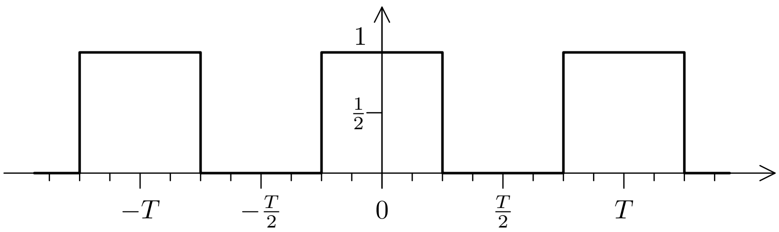
$$X(f) = \int_{-\infty}^{\infty} x(t)\,\mathrm{e}^{-2\pi \mathrm{j} f t}\,\mathrm{d}t$$

of the following two functions:

(*i*) $x(t) = c_T(t)$, where $c_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$ [3 marks]

(*ii*) $x(t) = r_b(t)$, where $r_b(t) = \begin{cases} 1 & \text{if } |t| < b \\ \frac{1}{2} & \text{if } |t| = b \\ 0 & \text{otherwise} \end{cases}$ [5 marks]

(*b*) Consider this periodic, binary, square-wave clock signal $p(t)$, with period T, duty cycle 0.5 and maximum amplitude 1:

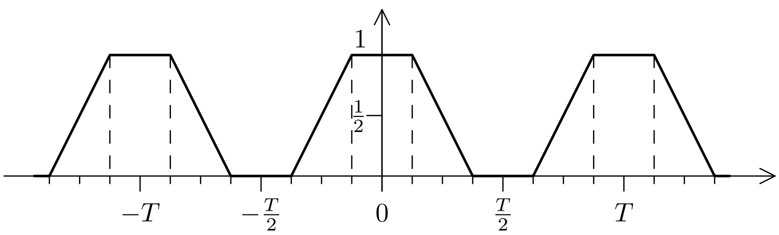


Show that its Fourier transform is

$$P(f) = \frac{1}{2}\delta(f) + \frac{1}{2\pi} \cdot \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{2k+1}{T}\right) \cdot \frac{(-1)^k}{k + \frac{1}{2}}.$$

Hint: Use the answers from part (*a*). [8 marks]

(*c*) Real-world digital signals need some time to transition between low and high. What is the Fourier transform of the periodic, trapezoid-wave clock signal $q(t)$, shown below, with period T and transition time $T/4$?



[4 marks]

4 Digital Signal Processing (MGK)

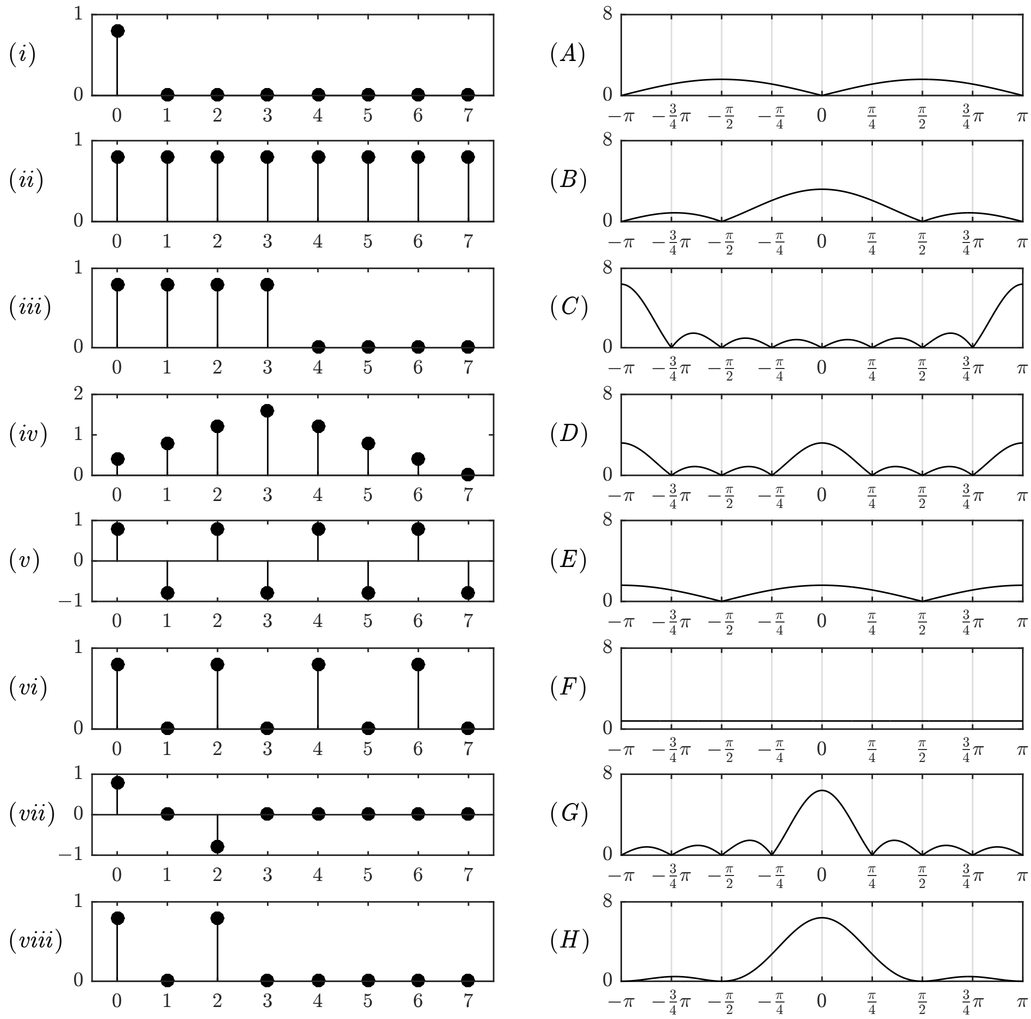
The discrete-time Fourier transform (DTFT) of a discrete sequence $\{x_n\}$ can be defined as

$$X(\mathrm{e}^{\mathrm{j}\omega}) = \sum_{n=-\infty}^{\infty} x_n \cdot \mathrm{e}^{-\mathrm{j}\omega n}$$

(*a*) If $\{x_n\}$ was the result of sampling a signal at sampling rate f_s and we want to know its DTFT at frequency f, what will be the corresponding value for ω? [2 marks]

(*b*) If $\{x_n\}$ has only real values and we know the value of $X(\mathrm{e}^{\mathrm{j}\pi/4})$, what is the value of $X(\mathrm{e}^{\mathrm{j}\pi\times 3.75})$? [2 marks]

(*c*) Each of the eight plots (*i*)–(*viii*) below shows real-valued samples $x_0, \ldots, x_7$ from a discrete sequence $\{x_n\}$, with $x_n = 0$ for $n < 0$ or $n > 7$. For each of these eight sequences, identify which of the eight plots (*A*)–(*H*) shows the magnitude $|X(\mathrm{e}^{\mathrm{j}\omega})|$ of the corresponding discrete-time Fourier transform. [8 × 2 marks]



5 Digital Signal Processing (MGK)

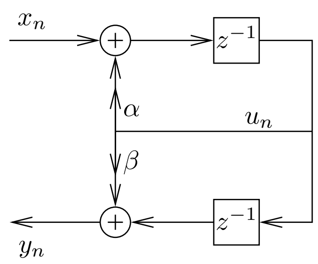
(*a*) You have been asked to design a long-wave radio receiver that can simultaneously monitor two radio signals at 60 ± 10 kHz and 100 ± 10 kHz, that is two stations with 20 kHz bandwidth each. Analog filters in your antenna amplifier suppress all signals outside those two bands.

What is the lowest possible sampling frequency that you can use for a *single* time-domain discrete sequence that records signals from these two bands simultaneously and unambiguously if you use

(*i*) base-band sampling; [2 marks]

(*ii*) IQ sampling. [2 marks]

(*b*) Consider the following digital filter with two multipliers:



(*i*) State the equations that define the elements of the output sequence $\{y_n\}$ and the intermediate sequence $\{u_n\}$ in terms of other values from $\{x_n\}$, $\{y_n\}$ or $\{u_n\}$. [4 marks]

(*ii*) Convert these equations into equivalent equations for the z-transforms $X(z)$, $Y(z)$ and $U(z)$ of these three discrete sequences, and then solve for $U(z)$ and $Y(z)$. [4 marks]

(*iii*) What is the z-transform $H(z)$ of the impulse response of this digital filter? [4 marks]

(*iv*) Draw the block diagram of an equivalent Direct Form I filter. [4 marks]

6 Digital Signal Processing (MGK)

(*a*) Consider a causal, order-2 digital filter with real-valued infinite impulse response sequence $h_0, h_1, h_2, \ldots$

(*i*) What is the z-transform $H(z)$ of this filter's impulse response? [2 marks]

(*ii*) Express $H(z)$ in terms of the locations c_1, c_2 of its two zeros and the locations d_1, d_2 of its two poles in $\mathbb{C}$. [4 marks]

(*iii*) Give a necessary condition for c_1, c_2, d_1, d_2 to ensure that $\{h_n\}$ has only real values. [4 marks]

(*iv*) If we operate that filter at sampling frequency f_s, what will its amplitude gain at frequency f be? [2 marks]

(*b*) A *notch filter* aims to suppress a single frequency f_c. One way of designing an order-2 notch filter, as in part (*a*), involves placing the zeros directly onto the unit circle, and the poles right next to them inside the unit circle, at distance $0 < \alpha < 1$ from 0:

$$c_1 = \mathrm{e}^{\mathrm{j}\omega}, \quad d_1 = \alpha \cdot c_1, \quad c_2 = \mathrm{e}^{-\mathrm{j}\omega}, \quad d_2 = \alpha \cdot c_2, \quad \text{with} \quad \omega = 2\pi f_\mathrm{c}/f_\mathrm{s}$$

(*i*) What is the z-transform of the impulse response of the resulting filter, written as a fraction of two polynomials of z^{-1}? [4 marks]

(*ii*) The *OxyMax* is a medical device designed in the United States. It processes a heart-beat signal with a sampling rate of $f_\mathrm{s} = 600$ Hz. It contains the following C function, which implements a notch filter, as in part (*b*)(*i*), to suppress in the input signal interference from the North American power grid at $f_\mathrm{c} = 60$ Hz:

```
double mains_notch(double sample) {
  static double x[4], y[4];
  static int n = 0;
  x[n&3] = sample;
  y[n&3] = sample + x[(n-1)&3] * b1 + x[(n-2)&3]
                  - y[(n-1)&3] * a1 - y[(n-2)&3] * a2;
  return y[n++&3];
}
```

The U.S. version initializes the constants used with `b1` $= -2\cos(\pi/5)$, `a1` $=$ `b1` $\times\ 0.9$ and `a2` $= 0.81$. What changed constant(s) will instead suppress the power-grid frequency at $f_\mathrm{c} = 50$ Hz for the European version? [4 marks]

5 Digital Signal Processing (MGK)

A discrete sequence $\{x_n\}$ can be converted into a continuous representation

$$\hat{x}(t) = t_\mathrm{s} \cdot \sum_{n=-\infty}^{\infty} \delta(t - n \cdot t_\mathrm{s}) \cdot x_n,$$

where t_s is the sampling period.

(*a*) State two characteristic properties of Dirac's δ function. [2 marks]

(*b*) Describe briefly how this representation helps to explain aliasing. [4 marks]

(*c*) Define three functions $h(t)$, such that convolving $\hat{x}(t)$ with $h(t)$ results in

(*i*) the output of an idealized analog-to-digital converter that holds the output voltage of each sample x_n for the time interval from $t = n \cdot t_\mathrm{s}$ until the next sample x_{n+1} arrives at time $t = (n+1) \cdot t_\mathrm{s}$; [4 marks]

(*ii*) linear interpolation of $\{x_n\}$; [4 marks]

(*iii*) reconstruction of a signal $x(t)$ that was sampled as $x_n = x(n \cdot t_\mathrm{s})$, assuming that the Fourier transform of $x(t)$ is zero at any frequency f with $|f|^{-1} \leq t_\mathrm{s}$ or $|f|^{-1} \geq 2t_\mathrm{s}$. [6 marks]

6 Digital Signal Processing (MGK)

Consider the discrete system

$$y_n = \sum_{i=0}^{\infty} x_{n-2i} \cdot \left(-\frac{1}{2}\right)^i$$

(*a*) Write down the first 8 samples of the impulse response of this filter. [2 marks]

(*b*) Provide the finite-difference equation of an equivalent recursive filter that can be implemented with not more than two delay elements. [4 marks]

(*c*) What is the z-transform $H(z)$ of the impulse response of this filter? [4 marks]

(*d*) Where are the zeros and poles of $H(z)$? [6 marks]

(*e*) We now operate this discrete system at sampling frequency $f_\mathrm{s} = 1$ MHz and feed it with input $x_n = \cos(2\pi f n / f_\mathrm{s})$. For which f (with $0 \le f \le f_\mathrm{s}/2$) will the peak amplitude of the output sequence $\{y_n\}$ be largest, and how large will it be? [4 marks]

5 Digital Signal Processing (MGK)

(*a*) Consider a digital filter with impulse response

$$h_i = 2\alpha \cdot \frac{\sin[2\pi(i - n/2)\alpha]}{2\pi(i - n/2)\alpha} \cdot w_i \quad \text{where} \quad w_i = \begin{cases} 1, & 0 \le i \le n \\ 0, & \text{otherwise} \end{cases}.$$

(*i*) What type of filter is this? [4 marks]

(*ii*) How are the sampling rate f_s at which this filter is operated and its -6 dB cut-off frequency f_c related to parameter α? [2 marks]

(*b*) In an open-source audio-effect library, you find a C routine for processing a recorded voice to sound like it came over an analog phone line:

```
#include <math.h>
#define N 512
#define PI 3.14159265358979323846
void phone_effect(double *x, double *y, int m)
{
  double w, p, f, g, h[N+1];
  int i, k;
  for (i = 0; i <= N; i++) {
    w = 0.54 - 0.46 * cos(2*PI*i/N);
    p = 2 * PI * (i-N/2) / 10;
    f = w * ((p == 0) ? 1 : sin(p)/p) / 5;
    p = 2 * PI * (i-N/2) / 100;
    g = w * ((p == 0) ? 1 : sin(p)/p) / 50;
    h[i] = f - g;
  }
  for (i = 0; i < m; i++) {
    y[i] = 0;
    for (k = 0; k <= N && k <= i; k++)
      y[i] += x[i - k] * h[k];
  }
}
```

The input array `x` and the output array `y` each hold `m` samples of an audio recording (mono) at sampling frequency $f_\mathrm{s} = 32$ kHz.

(*i*) Explain in detail what operation is implemented here (e.g., type of filter, order, cut-off frequency) and how it has been constructed. [8 marks]

(*ii*) You want to use this algorithm on audio recordings with a sampling rate of 48 kHz. What do you have to change in the source code to ensure that the audible effect remains the same? [6 marks]

6 Digital Signal Processing (MGK)

BBC Radio Cambridgeshire broadcasts a radio signal on a carrier frequency of 1026 kHz with a (double-sided) bandwidth of 10 kHz. You connect a long wire (antenna) via an amplifier and bandpass filter (0.5–2.0 MHz) to an analog-to-digital converter (ADC) with a sampling frequency of 5 MHz.

(*a*) You record $n = 500$ consecutive samples $(x_0, x_1, \ldots, x_{499})$ from the analog-to-digital converter output and calculate the Discrete Fourier Transform (DFT)

$$X_k = \sum_{i=0}^{n-1} x_i \cdot \mathrm{e}^{-2\pi \mathrm{j} \frac{ik}{n}}$$

(*i*) For which index value(s) k do you expect $|X_k|$ to best indicate the received signal strength of this radio station? [4 marks]

(*ii*) What preprocessing step would improve this indication? [4 marks]

(*iii*) What redundancy do you expect to find in the DFT output vector X, considering that the input signal is real-valued? [2 marks]

(*iv*) Explain a technique that exploits this redundancy to calculate this real-valued DFT more efficiently. (You can assume that an FFT implementation is already available and that $n = 512$ is used instead.) [4 marks]

(*b*) You want to record the output of this radio station for later analysis, but you do not yet know how it was modulated. How can you convert the ADC output sequence $\{x_i\}$ such that the resulting sequence encodes efficiently what is happening in the frequency range 1021–1031 kHz, with as low a sample rate as possible? [4 marks]

(*c*) You finally learn that the signal recorded in (*b*) was an amplitude-modulated positive mono audio signal. How can you demodulate it? [2 marks]

5 Digital Signal Processing (MGK)

(*a*) Make the following statements correct by changing one word or number. (Negating the sentence is not sufficient.)

(*i*) The z-transform of a sequence shows on the unit circle its discrete-time cosine transform. [1 mark]

(*ii*) Delaying a sequence by two samples corresponds in the z-domain to multiplication with z^2. [1 mark]

(*b*) Consider a causal digital IIR filter of order 2, operated at a sampling frequency of 48 kHz, where the impulse response $\{h_n\}$ has (for $n > 2$) the shape of a sine wave of frequency 8 kHz (amplitude and phase do not matter).

(*i*) Where in the z domain can you place two zeros and two poles to achieve such an impulse response $\{h_n\}$ in the time domain? [4 marks]

(*ii*) Write down the z transform of $\{h_n\}$ as a rational function (with those zeros and poles). [6 marks]

(*iii*) Provide the constant-coefficient difference equation that describes the time-domain behaviour of that filter. [4 marks]

(*iv*) How can you use such a filter design to digitally generate an 8 kHz sinewave sampled at 48 kHz with very little computational effort? [4 marks]

6 Digital Signal Processing (MGK)

(*a*) What can you say about the Fourier transform $X(f)$ if

(*i*) $x(t)$ is real; [2 marks]

(*ii*) $x(t) = -x(-t)$. [2 marks]

(*b*) Give the result of the Fourier transform $X(f) = \int_{-\infty}^{\infty} x(t)\, \mathrm{e}^{-2\pi \mathrm{j} f t}\, \mathrm{d}t$, using Dirac's delta where appropriate, of

(*i*) $x(t) = 1$; [1 mark]

(*ii*) $x(t) = \cos(2\pi t)$; [2 marks]

(*iii*) $x(t) = \mathrm{rect}(t)$; [2 marks]

(*iv*) $x(t) = [\frac{1}{2} + \frac{1}{2} \cdot \cos(2\pi t)] \cdot \mathrm{rect}(t)$. [3 marks]

(*c*) When is a random sequence $\{x_n\}$ called a "white noise" signal? [2 marks]

(*d*) Consider an n-dimensional random vector variable $\mathbf{X}$.

(*i*) How is its covariance matrix defined? [2 marks]

(*ii*) How can you change its representation without loss of information into a random vector of equal dimensionality in which all elements are mutually uncorrelated? [4 marks]

4 Digital Signal Processing (MGK)

(*a*) Make the following statements correct by changing one word or number. (Negating the sentence is not sufficient.)

(*i*) A square-summable sequence is also called power signal. [1 mark]

(*ii*) Adding together sine waves of the same frequency always results in another sine wave of the same phase. [1 mark]

(*iii*) The Fourier transform of a Dirac comb is a Dirac impulse. [1 mark]

(*iv*) 60 dBm = 1 W [1 mark]

(*b*) Before resampling a digital image at a quarter of its original resolution, you want to apply an anti-aliasing low-pass filter.

(*i*) If you apply a 1-dimensional filter with impulse response $\{h_n\}$ both horizontally and vertically to image pixels $I_{x,y}$, what are the resulting filtered pixel values $\tilde{I}_{x,y}$? [4 marks]

(*ii*) What would be the discrete impulse response $\{h_n\}$ of an ideal low-pass filter for this application, if its length were of no concern? [4 marks]

(*iii*) You decide to truncate the impulse response $\{h_n\}$ at its second zero-crossing on each side, resulting in a new impulse response $\{\bar{h}_n\}$. In the frequency domain, this results in $\bar{H} = H * T$ for what function T? [4 marks]

(*iv*) In order to make the frequency-domain response of your filter $\{\bar{h}_n\}$ smoother, you convolve it in the frequency domain with a rectangular pulse, the width of which is twice the distance between the zero crossings of T. What does the resulting time-domain impulse response $\{\breve{h}_n\}$ look like? [4 marks]

6 Digital Signal Processing (MGK)

The *Purpletoe* standard for trouser-area networking uses a radio signal with a bandwidth of less than 1 MHz. The carrier frequency is $f_c(k) = (2400 + 2k)$ MHz, where $k \in \{1, 2, 3, \ldots, 40\}$ is the channel number. Consider a receiver design in which the antenna signal is first multiplied with a sine wave of *fixed* frequency f_m, is then band-pass filtered to eliminate frequencies outside the range 1 MHz to 100 MHz, and is finally sampled by an analogue-to-digital converter with sampling frequency f_s for further digital processing.

(*a*) What is the largest set of frequencies from which f_m can be chosen such that no information is lost from any of the 40 channels? [4 marks]

(*b*) Which of the combinations of f_m and f_s that preserve all information from all 40 channels in the sampled output has the lowest sampling frequency f_s, assuming there is no signal outside these channels? [4 marks]

(*c*) To make eavesdropping more difficult, *Purpletoe* transmitters hop several times each second from one channel to another, in a secret pseudo-random order that is cryptographically pre-agreed and shared only with intended receivers. Consider for your receiver a special eavesdropping mode that exploits aliasing such that transmissions of a data packet using different channel numbers k all look the same after sampling (assuming that there is only a single transmitter in range). Which combination of f_s and f_m achieves that, and how? [8 marks]

(*d*) Cost pressures force you to use a cheaper circuit that multiplies the radio signal with a square wave of frequency f_m, instead of a sine wave. How does this affect the design of your receiver? [4 marks]

7 Digital Signal Processing (MGK)

(*a*) Make the following statements correct by changing one word or number. (Negating the sentence is not sufficient.)

(*i*) An absolutely summable discrete sequence will have in the corresponding z-transform plane at $z = 1$ a positive value. [1 mark]

(*ii*) A memory-less system depends only on the next input value. [1 mark]

(*b*) Define the convolution operator on discrete sequences. [2 marks]

(*c*) Prove that convolution of discrete sequences is an associative operation. [6 marks]

(*d*) Given samples $x_n = x(t_{\mathrm{s}} \cdot n)$ for all integers n, where $x(t)$ is a continuous signal whose Fourier transform has non-zero values only at frequencies f with $f_{\mathrm{l}} < |f| < f_{\mathrm{h}}$,

(*i*) under which condition can the original waveform $x(t)$ be reconstructed; [4 marks]

(*ii*) and how can this be done? [6 marks]

6 Digital Signal Processing (MGK)

While reverse-engineering a radio receiver, you find in its firmware the following two discrete systems implemented:

$$
\begin{aligned}
y_n &:= x_n \mathrm{e}^{\mathrm{j}\pi n/2} \\
z_n &:= \sum_{k=-4000}^{4000} y_{n-k-4000} \times 10^{-3} \operatorname{sinc}(k/10^3) \times \left(0.54 - 0.46 \cos\left(2\pi \frac{k+4000}{8000}\right)\right)
\end{aligned}
$$

The discrete sequence $\{x_n\}$ emerges from an analog-to-digital converter operating at sampling frequency $f_s = 240$ kHz, whose input is connected via a 100 kHz low-pass filter and linear amplifier directly to a radio antenna.

(a) Explain the function of *both* discrete systems in the frequency domain and their main parameters (e.g., type of filter, cutoff frequency, type of window). [12 marks]

(b) In approximately which frequency range will antenna signals substantially influence the resulting sequence $\{z_n\}$? [4 marks]

(c) Will the subsequent application of the discrete system

$$b_n := z_{n\times 500}$$

cause aliasing, and why? [4 marks]

12 Digital Signal Processing (MGK)

(*a*) Make the following statements correct by changing one word or number. (Negating the sentence is not sufficient.)

(*i*) In the stopband, a filter design approximates a gain of -1. [1 mark]

(*ii*) For infinite sequences the z-transform always converges across the entire complex plane. [1 mark]

(*iii*) The Barlett window is the product of a rectangular window and a raised cosine function. [1 mark]

(*iv*) Multiplying two complex variables can be implemented with two real-valued multiplications and five real-valued additions. [1 mark]

(*v*) As a continuous signal is sampled, its Fourier spectrum becomes non-linear. [1 mark]

(*b*) Briefly explain

(*i*) the zigzag ordering of DCT coefficients in JPEG; [3 marks]

(*ii*) the difference between I-, P- and B-frames in MPEG; [3 marks]

(*iii*) the relationship between RGB and YCrCb colour coordinates. [4 marks]

(*c*) A 300 Hz sine wave is sampled at 1000 Hz. This discrete sequence is then multiplied, sample by sample, with the discrete sequence

$$\ldots, 0, +1, 0, -1, 0, +1, 0, -1, 0, +1, 0, -1, \ldots$$

Which frequencies appear in the Fourier transform of the result? [5 marks]

11 Digital Signal Processing (MGK)

(*a*) What is the Fourier transform of a rectangular pulse of amplitude A and duration $d > 0$, centred around $t = 0$? [4 marks]

(*b*) Calculate the Fourier transform of the triangular pulse

$$\Lambda(t) = \begin{cases} 1 - |t|, & \text{for } |t| < 1 \\ 0, & \text{otherwise} \end{cases}$$

[*Hint:* Think of $\Lambda(t)$ as the result of a convolution.] [4 marks]

(*c*) A 2 kHz sine wave is sampled at 12 kHz. The resulting values are later converted back into a continuous signal using *linear interpolation*.

(*i*) At what other frequencies besides 2 kHz is there signal energy in the resulting continuous waveform? [4 marks]

(*ii*) Consider among those other components the one with the lowest frequency. By what factor is its voltage lower compared to the 2 kHz component? [4 marks]

(*iii*) Your colleague records with a PC soundcard at 44.1 kHz sampling frequency 1024 samples of the continuous waveform, loads these into MATLAB as vector `x` and then attempts to plot an amplitude spectrum with the command

```
plot(real(fft(x)));
```

Name two problems that need to be fixed in this command before the resulting plot is likely to agree with the result of (*c*)(*ii*). [4 marks]

11 Digital Signal Processing (MGK)

(*a*) A radio system outputs signals with frequency components only in the range 2.5 MHz to 3.5 MHz. The analog-to-digital converter that you want to use to digitize such signals can be operated at sampling frequencies that are an integer multiple of 1 MHz. What is the *lowest* sampling frequency that you can use without destroying information through aliasing? [5 marks]

(*b*) Consider a digital filter with an impulse response for which the z-transform is

$$H(z) = \frac{(z+1)^2}{(z-0.7-0.7\mathrm{j})(z-0.7+0.7\mathrm{j})}$$

(*i*) Draw the location of zeros and poles of this function in relation to the complex unit circle. [2 marks]

(*ii*) If this filter is operated at a sampling frequency of 48 kHz, which (approximate) input frequency will experience the lowest attenuation? [2 marks]

(*iii*) Draw a direct form I block-diagram representation of this digital filter. [5 marks]

(*c*) Make the following statements correct by changing one word or number. (Negating the sentence is not sufficient.)

(*i*) Statistical independence implies negative covariance. [1 mark]

(*ii*) Group 3 MH fax code uses a form of arithmetic coding. [1 mark]

(*iii*) Steven's law states that rational scales follow a logarithmic law. [1 mark]

(*iv*) The Karhunen-Loève transform is commonly approximated by the z-transform. [1 mark]

(*v*) 40 dB corresponds to an 80× increase in voltage. [1 mark]

(*vi*) The human ear has about 480 critical bands. [1 mark]

10 Digital Signal Processing (MGK)

(*a*) The DAUB4 wavelet transform involves a pair of 4-point FIR filters.

(*i*) Explain the properties that these filters are designed to have and provide a system of equations that defines the two impulse responses accordingly. [8 marks]

(*ii*) Explain briefly how this filter pair is used in the wavelet transform. [4 marks]

(*b*) Consider a digital radio designed to receive all signals in the frequency range 90–105 MHz. Its antenna amplifier includes a bandpass filter that eliminates any signals outside this frequency range. The filtered antenna signal is directly fed into a digital-to-analogue converter, such that all subsequent demodulation steps can be performed in software.

(*i*) What is the lowest sampling frequency that can be used without risking loss of information due to aliasing? Explain briefly why. [5 marks]

(*ii*) If the resulting discrete sequence were turned into a continuous baseband signal through sinc interpolation, what relationship would there be between the spectra of the input and output signal? In particular, what would a 94 MHz sine-wave antenna signal be converted into? [3 marks]

10 Digital Signal Processing (MGK)

(*a*) Write an efficient microcontroller program (pseudo code) that outputs a continuous sine wave of frequency $f = 440$ Hz with values y_n in the range -1 to 1 at a sampling frequency $f_s = 32$ kHz. The programming language you have available lacks complex-number arithmetic, the runtime environment offers only basic floating-point arithmetic (i.e., no trigonometric functions), addition is much faster than multiplication, and there is insufficient memory to store a precomputed waveform. [10 marks]

(*b*) The discrete sequence $y_n = \cos(2\pi n f_1/f_s) + A \cdot \cos(2\pi n f_2/f_s)$ is fed into a (hypothetical) digital-to-analogue converter that outputs a constant voltage $y(t) = y_n$ during the time interval $n/f_s \leq t < (n+1)/f_s$ for all integers n.

(*i*) Explain how this behaviour of the digital-to-analogue converter affects the amplitude spectrum of the resulting signal. [5 marks]

(*ii*) What amplitude A has to be chosen for the second term such that the resulting amplitude spectrum shows equally high peaks at both $f_1 = 1$ kHz and $f_2 = 2$ kHz if the sampling frequency is $f_s = 6$ kHz. [5 marks]

10 Digital Signal Processing (MGK)

(*a*) Consider a software routine that converts and records the audio samples received in a digital telephone network call (8 kHz sampling frequency, 8 bit/sample) into a WAV file (8 kHz sampling frequency, 16 bit/sample, uniform quantisation). Your colleague attempted to write a very simple conversion routine for this task, but the resulting audio is very distorted.

(*i*) Name two variants of the method used for quantising the amplitude of audio samples in digital telephone networks and explain one of them. [4 marks]

(*ii*) Your colleague's routine right-pads each 8-bit data word from the telephone network with eight additional least-significant zero bits to obtain 16 bit values. Explain how this distorts the signal by discussing which frequencies could appear at the output when the incoming telephone signal consists of a pure 1 kHz sine tone. [4 marks]

(*b*) A real-valued discrete random sequence $\{x_i\}$ is fed into a linear time-invariant filter with impulse response $h_0 = 1$, $h_3 = 1$, and $h_i = 0$ for all other i. We observe for the resulting output sequence $\{y_i\}$ the expected value

$$\mathcal{E}(y_{i+k} \cdot x_i) = \begin{cases} 1 & \text{for } k = -1 \\ 2 & \text{for } k = 0 \\ 1 & \text{for } k = 1 \\ 1 & \text{for } k = 2 \\ 2 & \text{for } k = 3 \\ 1 & \text{for } k = 4 \\ 0 & \text{otherwise} \end{cases}$$

What is the value of the autocorrelation sequence $\{\phi_{xx}(k)\}$? [4 marks]

(*c*) The $YCrCb$ colour encoding is used in many image compression methods.

(*i*) How is it defined and why is it used? [4 marks]

(*ii*) Is the conversion from 3×8-bit RGB to 3×8-bit $YCrCb$ coordinates fully reversible? Why? [4 marks]

10 Digital Signal Processing (MGK)

Consider a software routine that converts the sampling rate of digital audio data from 8 kHz to 48 kHz, without changing the represented sound. It reads an input sequence $\{x_i\}$ and produces an output sequence $\{y_i\}$. The routine first inserts five samples of value 0 between each consecutive pair of input samples. This results in a new intermediate sequence $\{x'_i\}$ with $x'_{6i} = x_i$ and $x'_{6i+k} = 0$ for all $k \in \{1, \dots, 5\}$. The sequence $\{x'_i\}$ is then low-pass filtered, resulting in $\{y_i\}$.

(*a*) How can the process of taking discrete-time samples $\{x_i\}$ from a continuous waveform $x(t)$ be modelled through a function $\hat{x}(t)$ that represents the sampling result but can still be analysed using the continuous Fourier transform? [2 marks]

(*b*) What effect does sampling with 8 kHz have on the Fourier spectrum of the signal? [2 marks]

(*c*) How and under what condition can this sampling process be reversed? [2 marks]

(*d*) Can $\hat{x}(t)$ also model another sampling process that results in the discrete sequence $\{x'_i\}$, and if so, what is its sampling frequency? [2 marks]

(*e*) How does the continuous spectrum associated with $\{x'_i\}$ relate to that of $\{x_i\}$? [2 marks]

(*f*) What purpose serves the low-pass filter that the routine applies? In particular, what would happen to a 1 kHz sine tone input if this filter were not applied and $\{y_i\} = \{x'_i\}$ were output instead? What cut-off frequency must the filter have? [5 marks]

(*g*) Provide a formula for calculating a 25-sample long causal finite impulse response $\{h_i\}$ of a low-pass filter suitable for this routine, based on the Hamming windowing function. [5 marks]

10 Digital Signal Processing (MGK)

(*a*) Characterise the systems below as linear/nonlinear, causal/noncausal, and time invariant/time varying:

(*i*) $y_n = ax_{3n-2}$ [2 marks]

(*ii*) $y_n = y_{n-1} + 6x_{n-2}$ [2 marks]

(*iii*) $y_n = y_{n-1} - x_{n+5} + x_{n-5}$ [2 marks]

(*iv*) $y_n = \frac{x_n}{x_{n-3}y_{n-2}}$ [2 marks]

(*v*) $y_n = x_n - \cos\left(\frac{\pi}{2}n\right)$ [2 marks]

(*b*) Consider the system $h : \{x_n\} \to \{y_n\}$ with $y_n - y_{n-1} = x_n - x_{n-3}$.

(*i*) Give the impulse response of this system. [2 marks]

(*ii*) Give one sine-wave input sequence of the form

$$x_n = a \cdot \sin(b \cdot n + c)$$

(with $a \neq 0, b \neq 0$) for which $y_n = 0$ for all n. [2 marks]

(*iii*) Express the system h as a rational function $H(z)$. [3 marks]

(*iv*) Determine the values $z \in \mathbb{C}$ for which $H(z) = 0$. [3 marks]

10 Information Theory and Coding (MGK)

[. . .]

(*d*) Briefly explain

(*i*) how 10 V is expressed in dBμV; [1 mark]

(*ii*) the YCrCb coordinate system. [4 marks]

10 Digital Signal Processing (MGK)

(*a*) You have designed a digital water-level display installed on the river Cam. A sensor measures the current height of a small floating ball once every minute. In order to reduce the fluctuations that small waves would otherwise cause in the displayed value, you implemented a digital filter $y_i = 0.8y_{i-1} + 0.2x_i$, where the x_i are the measured and the y_i are the displayed water levels.

(*i*) What type of filter is this? [2 marks]

(*ii*) The standard deviation caused by small waves in the measurements is 30 mm. There is no measurable correlation between these added noise values. Calculate the standard deviation caused by small waves in the displayed water levels. [8 marks]

(*b*) Let H be a digital low-pass filter with finite impulse response $h_0, h_1, \ldots, h_7$. Let f_s be the sampling frequency. Give the impulse response $h'_0, h'_1, \ldots, h'_7$ of a filter H' with frequency response $|H'(f)| = |H(f_\mathrm{s}/2 - f)|$. [4 marks]

(*c*) A programmer cuts a block out of a digitized sound signal and applies the Discrete Fourier Transform to estimate its spectral power distribution.

(*i*) What effect distorts the resulting power spectrum? [3 marks]

(*ii*) Describe briefly one technique to reduce these distortions. [3 marks]

8 Information Theory and Coding (MGK)

[. . .]

(*c*) Explain briefly:

(*i*) sensation limit; [1 mark]

(*ii*) critical band; [1 mark]

(*iii*) Bark scale. [1 mark]

(*d*) Which different aspects of perception do Weber's law and Steven's law model? [2 marks]

10 Information Theory and Coding (MGK)

[. . .]

(*c*) You are asked to compress a collection of files, each of which contains several thousand photographic images. All images in a single file show the same scene. Everything in this scene is static (no motion, same camera position, etc.) except for the intensity of the five light sources that illuminate everything. The intensity of each of the five light sources changes in completely unpredictable and uncorrelated ways from image to image. The intensity of each pixel across all photos in a file can be described as a linear combination of the intensity of these five light sources.

(*i*) Which one of the five techniques *discrete cosine transform*, *μ-law coding*, *2-D Gabor transform*, *Karhunen-Loève transform* and *Golomb coding* would be best suited to remove redundancy from these files, assuming your computer is powerful enough for each? [1 mark]

(*ii*) Explain briefly this transform and why it is of use here. [4 marks]