Complexity Theory

Lecture 4: Reductions

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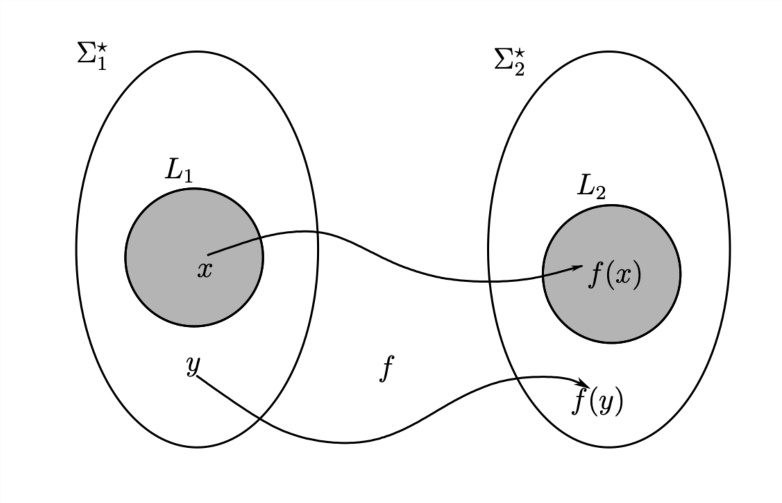
Recap

- Goal: Chart a landscape of complexity classes
- $\mathcal{P}$ captures polynomial-time computation
- $\mathcal{NP}$ captures polynomial-time verification
- The million dollars question: is $\mathcal{P} \neq \mathcal{NP}$?
- Natural problems outside of $\mathcal{NP}$?

First superpower of complexity theory: solving one problem using another!

Reductions



Reductions

Given two languages $L_1 \subseteq \Sigma_1^\star$, and $L_2 \subseteq \Sigma_2^\star$,

A *reduction* of L_1 to L_2 is a *computable* function

$$f : \Sigma_1^\star \to \Sigma_2^\star$$

such that for every string $x \in \Sigma_1^\star$,

$$f(x) \in L_2 \text{ if, and only if, } x \in L_1$$

What is missing here?

Resource Bounded Reductions

If f is computable by a polynomial time algorithm, we say that L_1 is *polynomial time reducible* to L_2.

$$L_1 \leq_P L_2$$

If f is also computable in $\mathrm{SPACE}(\log n)$, we write

$$L_1 \leq_L L_2$$

Why do we use the $\leq$ notation?

Reductions: an alternative perspective

If $L_1 \leq_P L_2$ we understand that L_1 is no more difficult to solve than L_2, at least as far as polynomial time computation is concerned.

That is to say,

If $L_1 \leq_P L_2$ *and* $L_2 \in \mathrm{P}$, *then* $L_1 \in \mathrm{P}$

We can get an algorithm to decide L_1 by first computing f, and then using the polynomial time algorithm for L_2.

Completeness

Reductions allow us to establish the *relative* complexity of problems, even when we cannot prove absolute lower bounds.

Cook (1972) first showed that there are problems in NP that are maximally difficult.

A language L is said to be NP-*hard* if for every language $A \in \mathrm{NP}$, $A \leq_P L$.

A language L is NP-*complete* if it is in NP and it is NP-hard.

What languages are NP-complete?

Cook-Levin Theorem: SAT is NP-complete

Cook and Levin independently showed that the language SAT of satisfiable Boolean expressions is NP-complete.

Recall that SAT is in NP. (why?)

It remains to show NP-hardness: for every language L in NP, there is a polynomial time reduction from L to SAT. (why is that possible?)

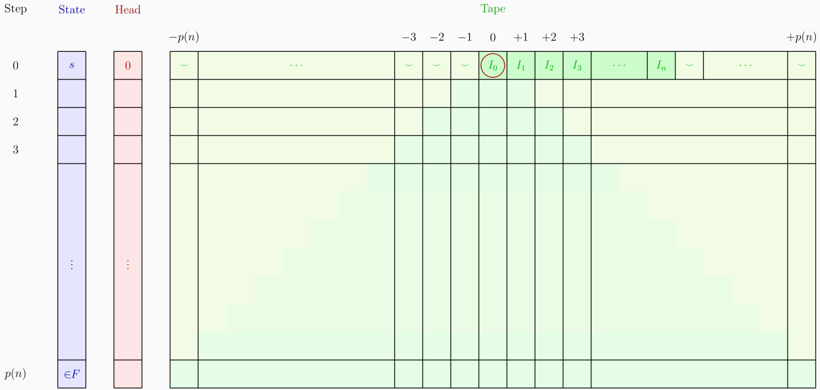
Since L is in NP, there is a nondeterministic Turing machine

$$M = (Q, \Sigma, s, \delta)$$

and a bound k such that a string x of length n is in L if, and only if, it is accepted by M within n^k steps.

Turing Machine Tableau

We need to give, for each $x \in \Sigma^\star$, a Boolean expression $f(x)$ which is satisfiable if, and only if, there is an accepting computation of M on input x.



$f(x)$ has the following variables:

$$
\begin{array}{ll}
S_{i,q} & \text{for each } i \leq n^k \text{ and } q \in Q \\
T_{i,j,\sigma} & \text{for each } i, j \leq n^k \text{ and } \sigma \in \Sigma \\
H_{i,j} & \text{for each } i, j \leq n^k
\end{array}
$$

Intuitively, these variables are intended to mean:

- $S_{i,q}$ – the state of the machine at time i is q.
- $T_{i,j,\sigma}$ – at time i, the symbol at position j of the tape is σ.
- $H_{i,j}$ – at time i, the tape head is pointing at tape cell j.

We now have to see how to write the formula $f(x)$, so that it enforces these meanings.

Initialization

The initial state is s, and the head is initially at the beginning of the tape.

$$S_{1,s} \wedge H_{1,1}$$

The initial tape contents are x

$$\bigwedge_{j \leq n} T_{1,j,x_j} \wedge \bigwedge_{n<j} T_{1,j,\sqcup}$$

Consistency

The head is never in two places at once

$$\bigwedge_{i} \bigwedge_{j} (H_{i,j} \to \bigwedge_{j' \neq j} (\neg H_{i,j'}))$$

The machine is never in two states at once

$$\bigwedge_{q} \bigwedge_{i} (S_{i,q} \to \bigwedge_{q' \neq q} (\neg S_{i,q'}))$$

Each tape cell contains only one symbol

$$\bigwedge_{i} \bigwedge_{j} \bigwedge_{\sigma} (T_{i,j,\sigma} \to \bigwedge_{\sigma' \neq \sigma} (\neg T_{i,j,\sigma'}))$$

Computation

The tape does not change except under the head

$$\bigwedge_{i} \bigwedge_{j} \bigwedge_{j' \neq j} \bigwedge_{\sigma} (H_{i,j} \wedge T_{i,j',\sigma}) \rightarrow T_{i+1,j',\sigma}$$

Each step is according to δ.

$$\begin{aligned} \bigwedge_{i} \bigwedge_{j} \bigwedge_{\sigma} \bigwedge_{q} &(H_{i,j} \wedge S_{i,q} \wedge T_{i,j,\sigma}) \\ &\rightarrow \bigvee_{\Delta} (H_{i+1,j'} \wedge S_{i+1,q'} \wedge T_{i+1,j,\sigma'}) \end{aligned}$$

where Δ is the set of all triples (q', σ', D) such that $((q, \sigma), (q', \sigma', D)) \in \delta$ and

$$j' = \begin{cases} j & \text{if } D = S \\ j - 1 & \text{if } D = L \\ j + 1 & \text{if } D = R \end{cases}$$

Finally, the accepting state is reached

$$\bigvee_i S_{i,\text{acc}}$$

CNF

A Boolean expression is in *conjunctive normal form* if it is the conjunction of a set of *clauses*, each of which is the disjunction of a set of *literals*, each of these being either a *variable* or the *negation* of a variable.

For any Boolean expression ϕ, there is an equivalent expression ψ in conjunctive normal form.

ψ can be exponentially longer than ϕ.

However, CNF-SAT, the collection of satisfiable CNF expressions, is NP-complete.

3SAT

A Boolean expression is in 3CNF if it is in conjunctive normal form and each clause contains at most 3 literals.

3SAT is defined as the language consisting of those expressions in 3CNF that are satisfiable.

3SAT is NP-complete, as there is a polynomial time reduction from CNF-SAT to 3SAT.

Composing Reductions

Polynomial time reductions are clearly closed under composition.

So, if $L_1 \leq_P L_2$ and $L_2 \leq_P L_3$, then we also have $L_1 \leq_P L_3$.

If we show, for some problem A in NP that

$$\text{SAT} \leq_P A$$

or

$$\text{3SAT} \leq_P A$$

it follows that A is also NP-complete.

Questions?