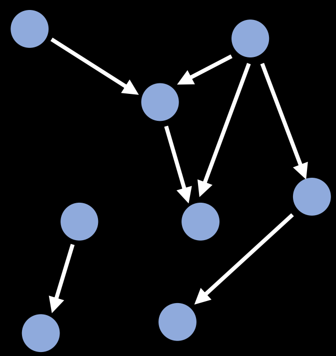
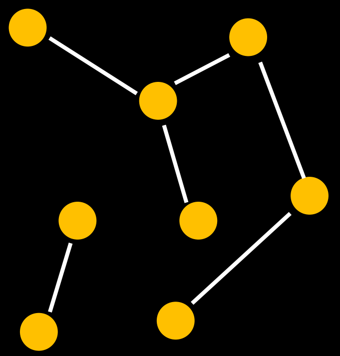
ALGORITHMS 2
SECTION 5

Graphs and path finding

directed graphs



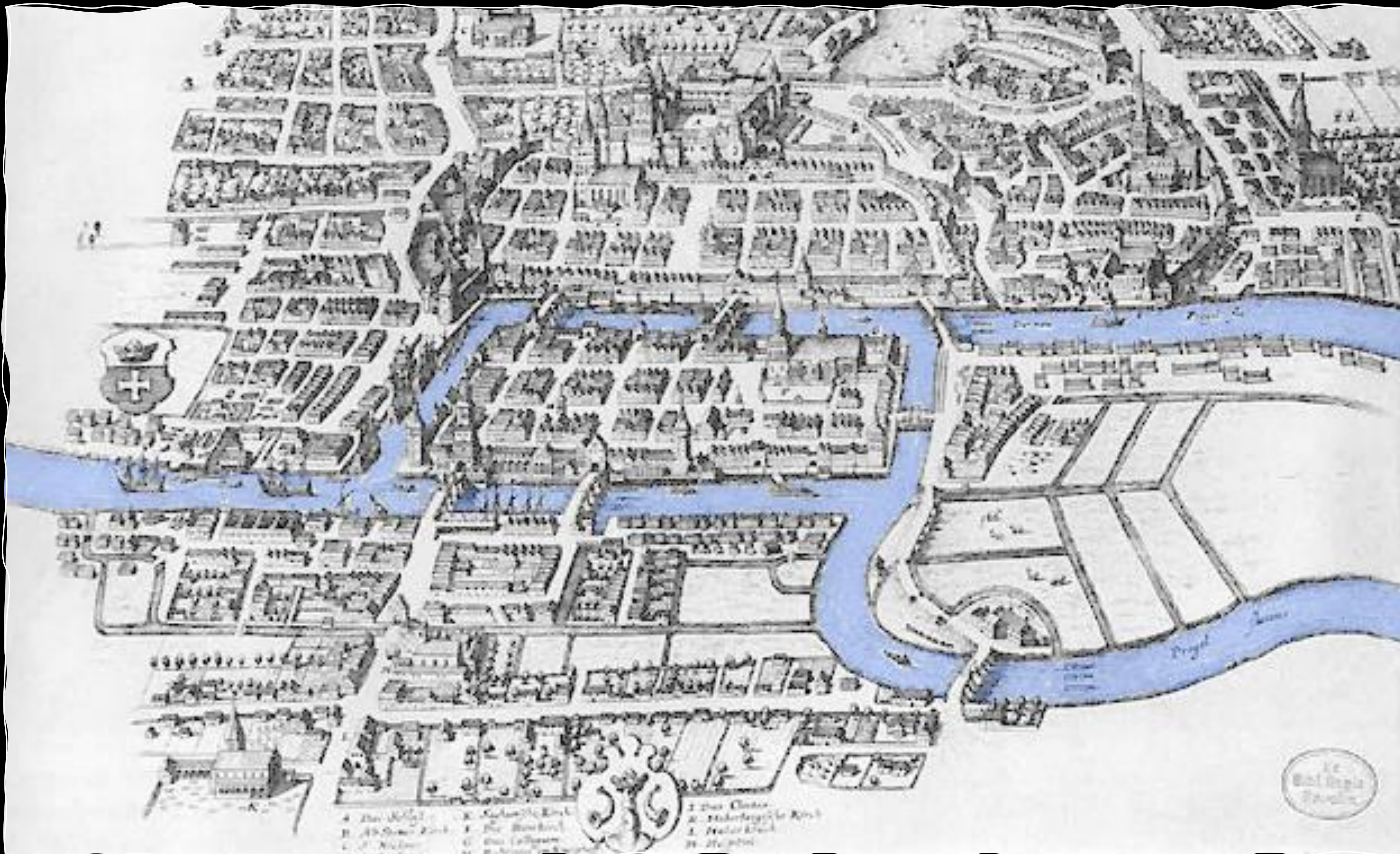
undirected graphs



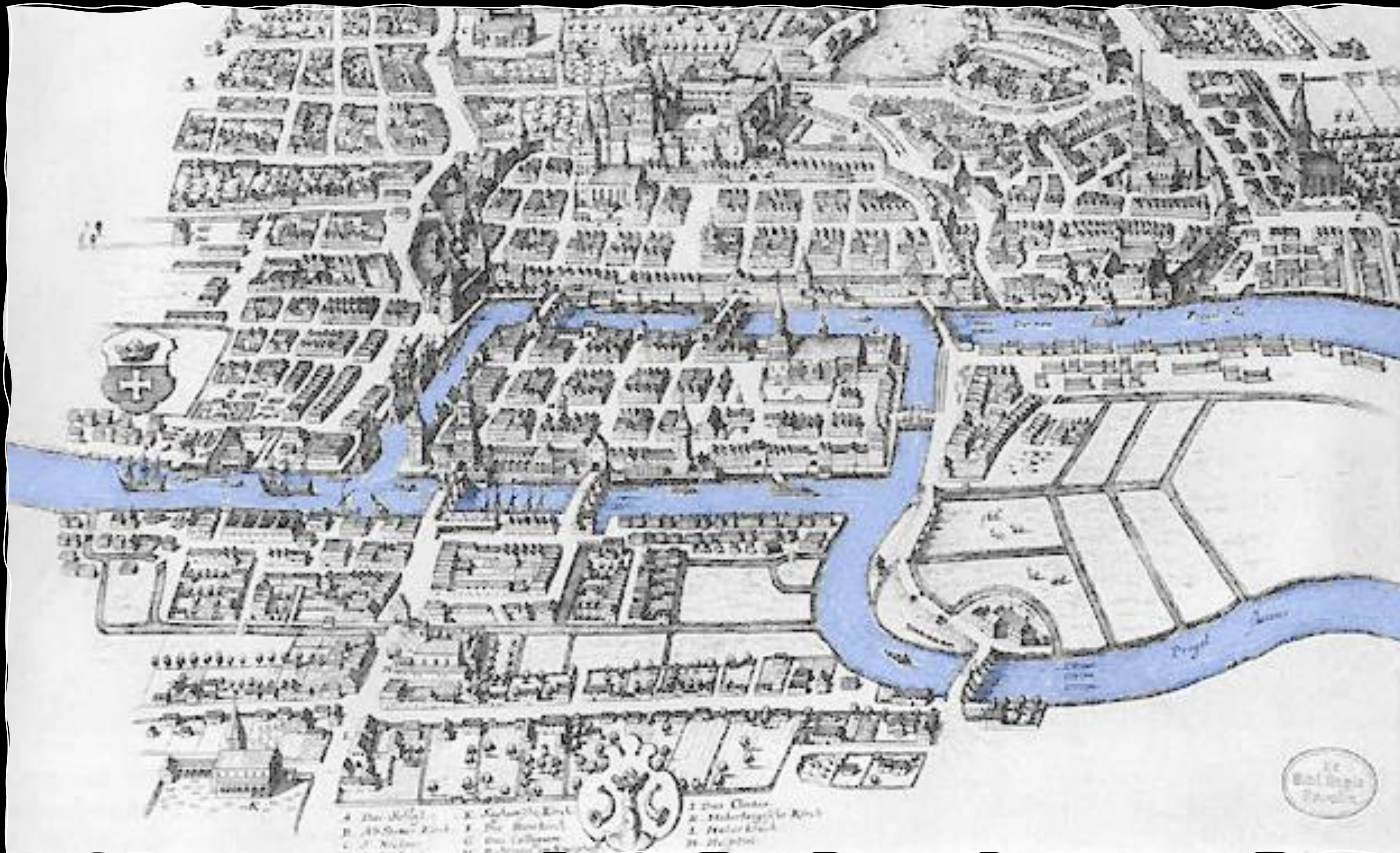
A directed graph is an ordered pair $G = (V, E)$ where V is a set ("vertices") and E is a relation on V ("edges").

same, except the relation is symmetric.

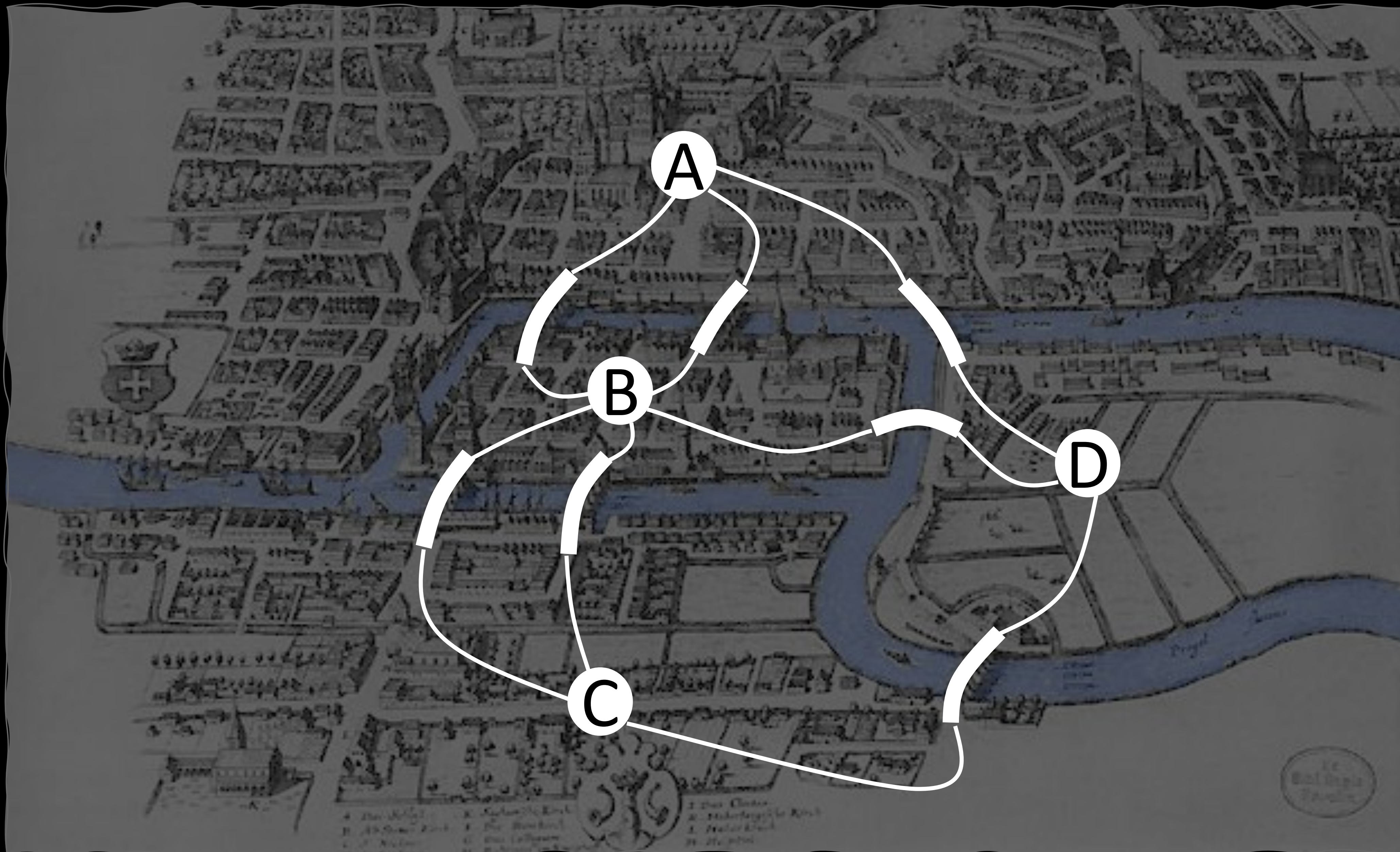
Konigsberga



“Can I go for a stroll around the city on a route that crosses each bridge exactly once?”

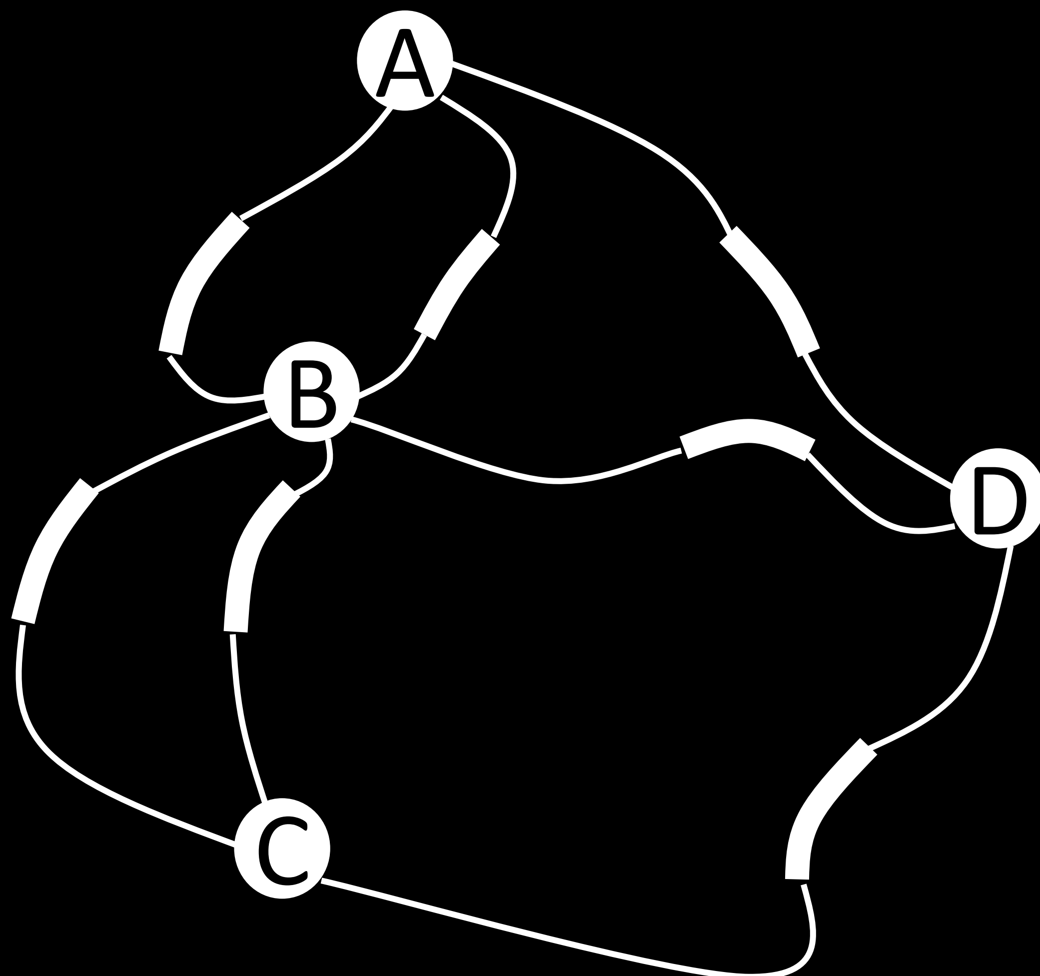


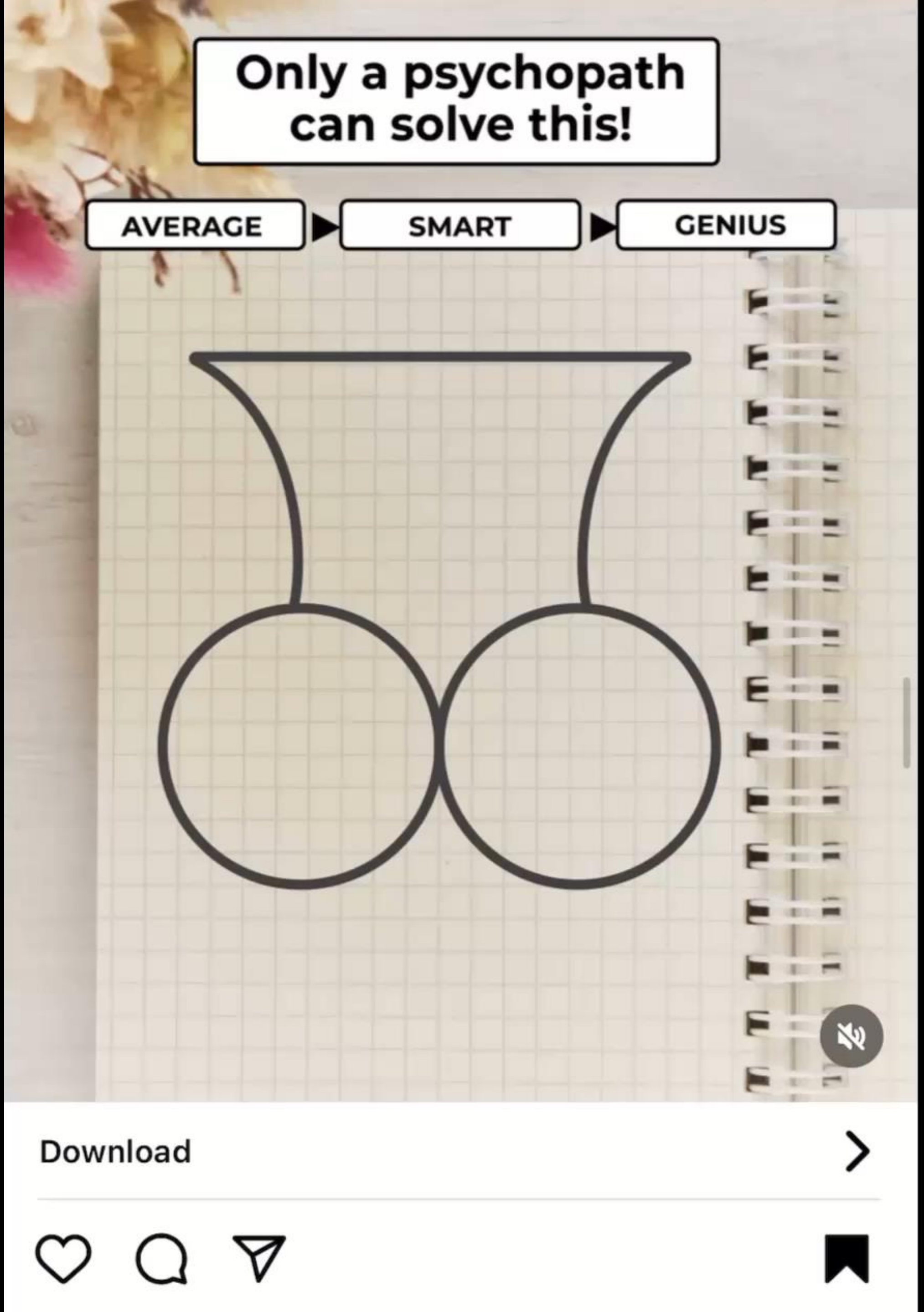
“Can I go for a stroll around the city on a route that crosses each bridge exactly once?”

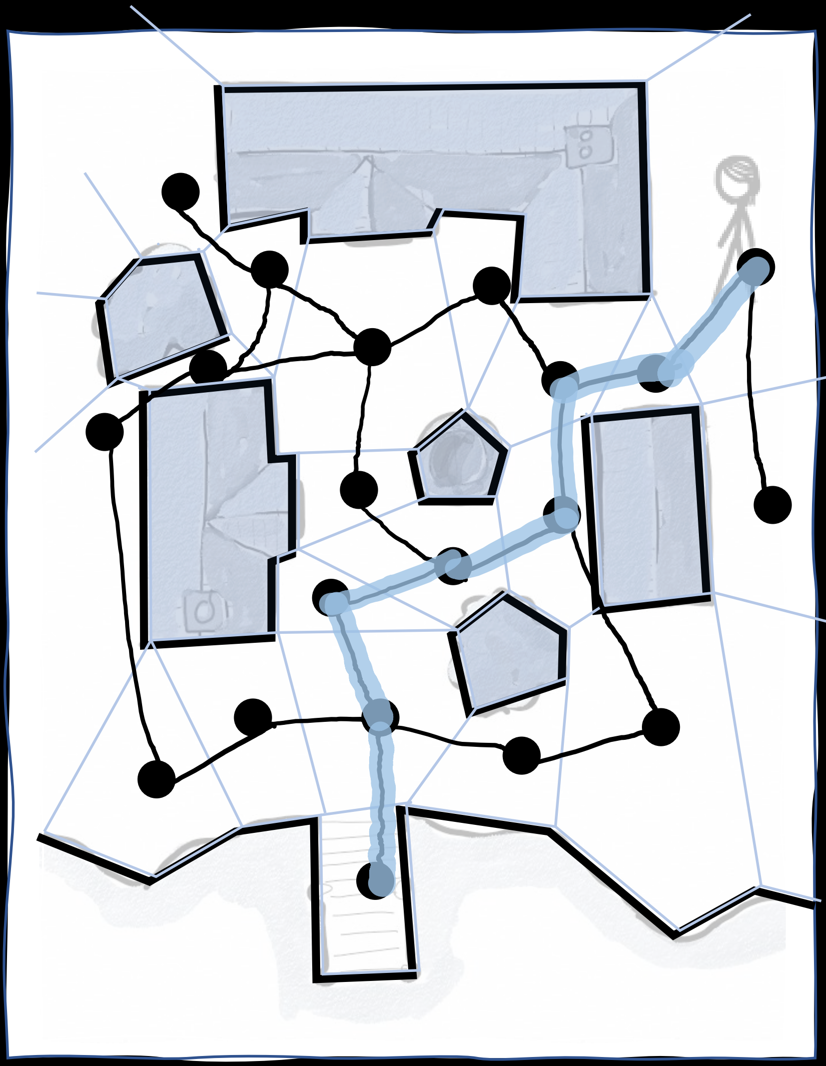


“Is there a path in which every edge appears exactly once?”

```
g = {A: [B,B,D],
     B: [A,A,C,C,D],
     C: [B,B,D],
     D: [A,B,C]}
```




Only a psychopath can solve this!
AVERAGE
SMART
GENIUS
Download



PATH-FINDING ALGORITHMS

How should this game agent navigate to the jetty?

1. Draw polygon boundaries around obstacles
2. Divide free space into convex polygons
3. Create a graph, with edges between adjacent polygons
4. Find a path on the graph
5. Draw this path in 2D coordinates on the map
   (easy, since we've used convex polygons)

Dwarf Fortress

How can we do path-finding at scale?

https://stackoverflow.blog/2021/12/31/700000-lines-of-code-20-years-and-one-developer-how-dwarf-fortress-is-built/

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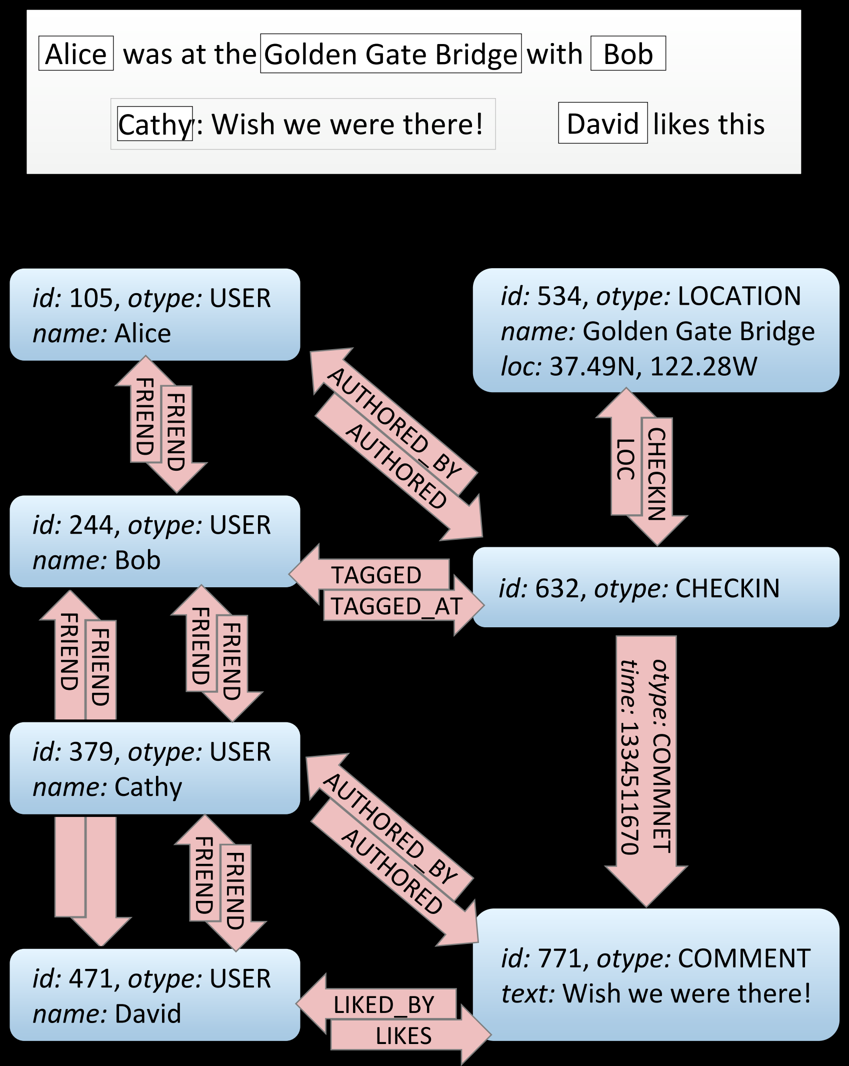


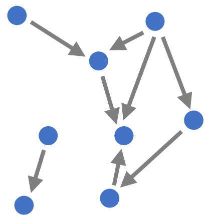
Q. Why did Facebook choose to make CHECKIN a vertex, rather than a USER→LOCATION edge?

- How fast will an epidemic of misinformation spread?
- At whom should I target my advertising?

Graph notation

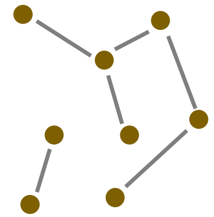
A graph consists of a set of vertices V, and a set of edges E.

directed graphs



$v_1 \rightarrow v_2$ is how we write
the edge from v_1 to v_2

undirected graphs



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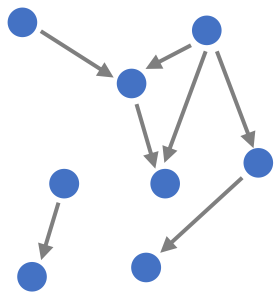
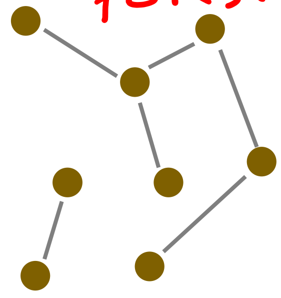
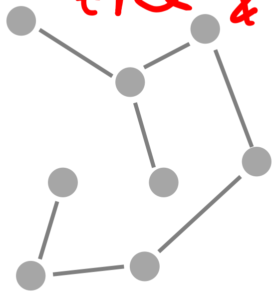


forest



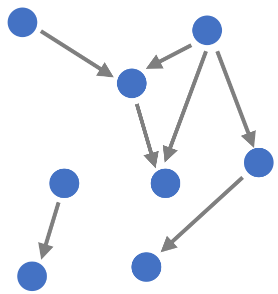
tree & forest

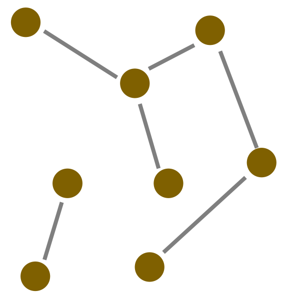


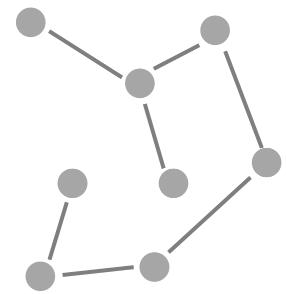
Which of these two graphs is a tree, which a forest?

- A *directed acyclic graph* (DAG) is a directed graph without any cycles

- A *forest* is an undirected acyclic graph
- A *tree* is a connected forest
- (An undirected graph is *connected* if for every pair of vertices there is a path between them)





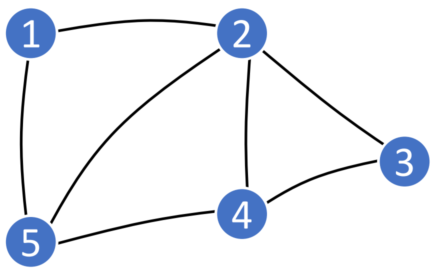


What’s wrong with my definitions for *path* and *cycle*?

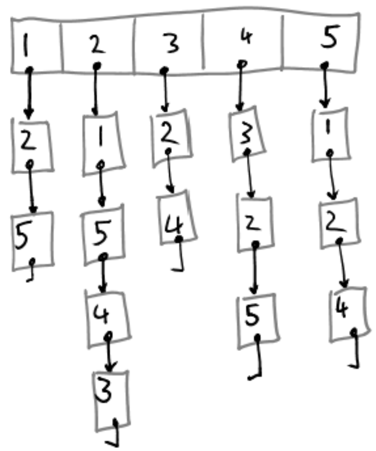
- A *directed acyclic graph* (DAG) is a directed graph without any cycles

- A *forest* is an undirected acyclic graph
- A *tree* is a connected forest
- (An undirected graph is *connected* if for every pair of vertices there is a path between them)

How we can store graphs, in computer code



Array of adjacency lists



Memory: O(V+E)

```
{1: [2,5],
 2: [1,5,4,3],
 3: [2,4],
 4: [3,2,5],
 5: [1,2,4]
 }
```

Note: when we write this we really mean $O(|V|+|E|)$, since V and E are sets.

Note: I'll gloss over for now the technical difficulty of defining big-O notation with two variables. We'll come back to it later.

For now, just read this as "we need |V| space for the vertex list, and |E| space in total for all the edge lists.

Adjacency matrix

	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	1	1
3	0	1	0	1	0
4	0	1	1	0	1
5	1	1	0	1	0

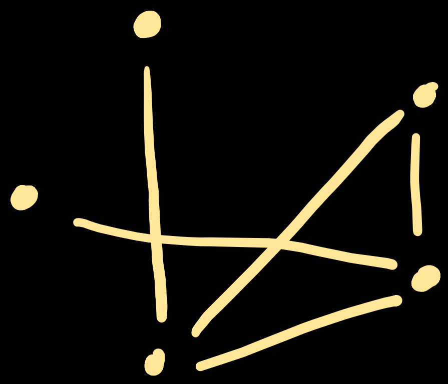
```
np.array([[0,1,0,0,1],
          [1,0,1,1,1],
          [0,1,0,1,0],
          [0,1,1,0,1],
          [1,1,0,1,0]])
```

Memory: $O(V^2)$

Mini-exercise

page 4

- What is the largest possible number of edges in an undirected graph with V vertices?
- and in a directed graph?
- What's the smallest possible number of edges in a tree with V vertices?



The next eight lectures

- Clever graph algorithms
- Performance analysis

- Proving correctness

- What we can model with graphs

IA Algorithms

Damon Wischik, Computer Laboratory, Cambridge University. Lent Term 2021

what was printed out earlier:
2021 notes

Contents

IA Algorithms 2

Damon Wischik, Computer Laboratory, Cambridge University. Lent Term 2024

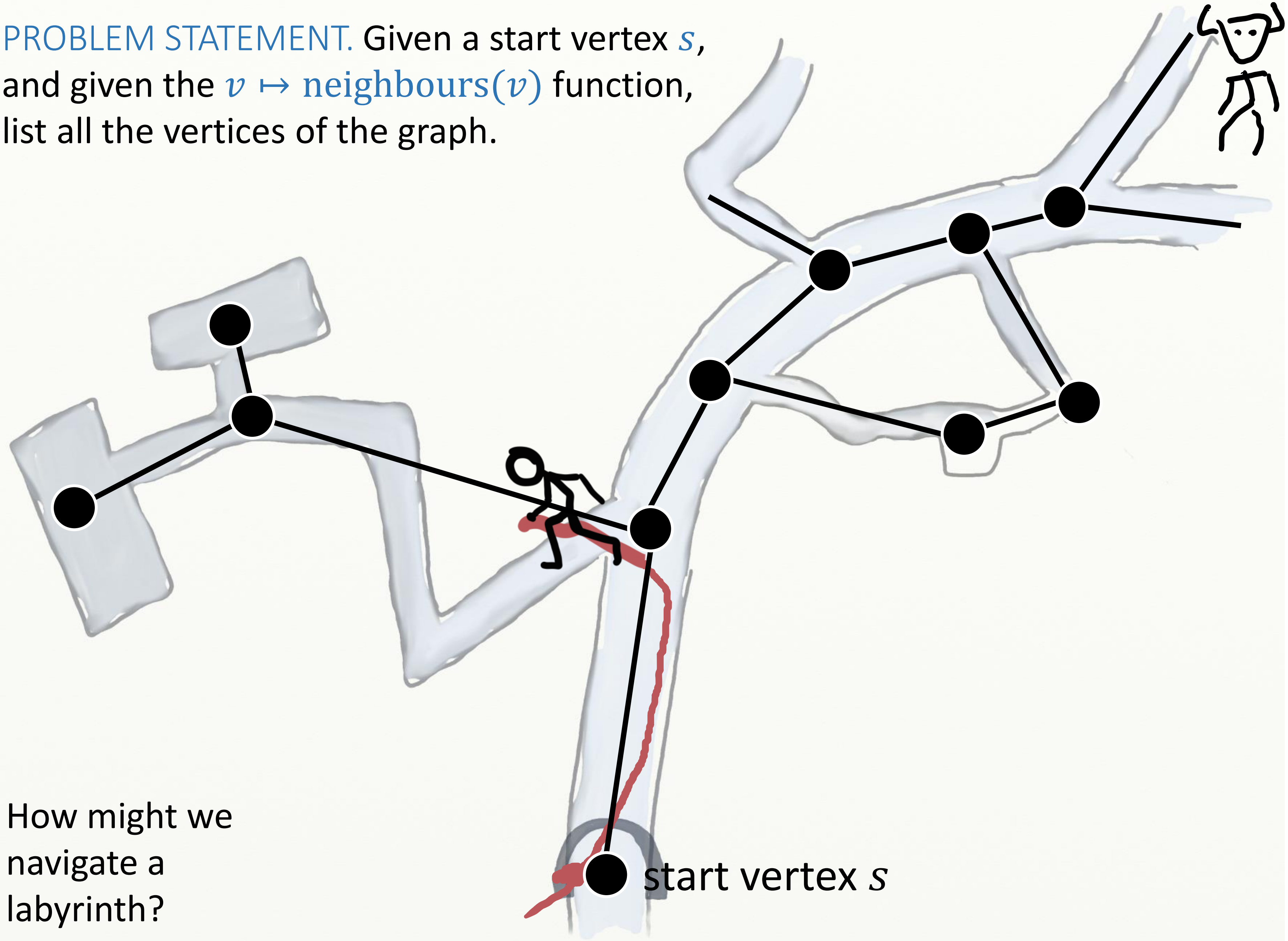
what's online, and will be ready to collect on Friday:
2024 notes

Contents

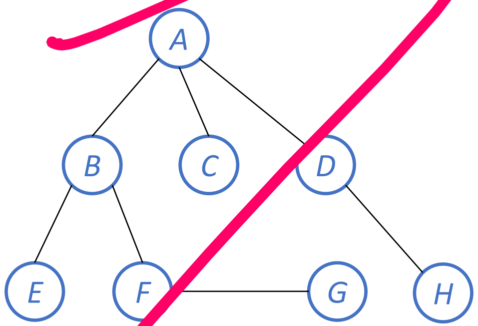
SECTION 5.2

Depth-first search

PROBLEM STATEMENT. Given a start vertex s, and given the $v \mapsto \text{neighbours}(v)$ function, list all the vertices of the graph.

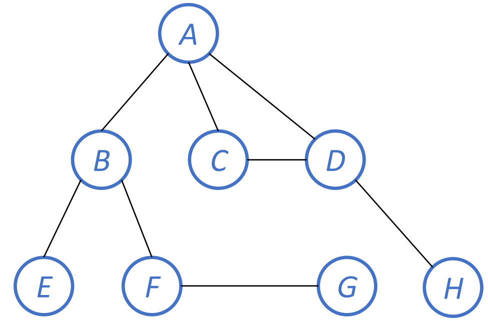


How might we navigate a labyrinth?



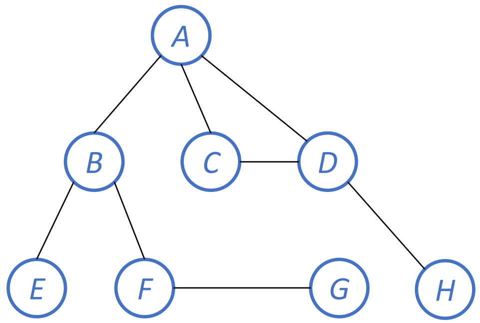

```
def visit(v):
    print("visiting", v)
    for w in v.neighbours:
        visit(w)

visit(A)
```

visiting A
visiting B
visiting A
visiting B
...
RecursionError:
maximum recursion depth exceeded




```
def visit_tree(v, v_parent):
    print("visiting", v, "from", v_parent)
    for w in v.neighbours:
        if w != v_parent:
            visit_tree(w, v)

visit_tree(D, None)
```

visiting D from None
visiting C from D
visiting A from C
visiting D from A
...
RecursionError:
maximum recursion depth exceeded




```
# visit all vertices reachable from s
def dfs_recurse(g, s):
    for v in g.vertices:          visited = set()
        v.visited = False
    visit(s)

def visit(v):
    v.visited = True              visited.add(v)
    for w in v.neighbours:
        if not w.visited:         v not in visited
            visit(w)
```

This algorithm needs to keep track of which vertices it's seen. It'd be cleaner to store this as a set — but we haven't yet done performance analysis of set operations, so in this code snippet I'm using a per-vertex attribute instead.

start vertex: D

```
dfs_recurse(g, D):
  visit(D):
    neighbours = [H, C, A]
    visit(H):
      neighbours = [D]
      don't visit D
      return from visit(H)
    visit(C)
      neighbours = [D, A]
      don't visit D
      visit(A):
        | ...
```

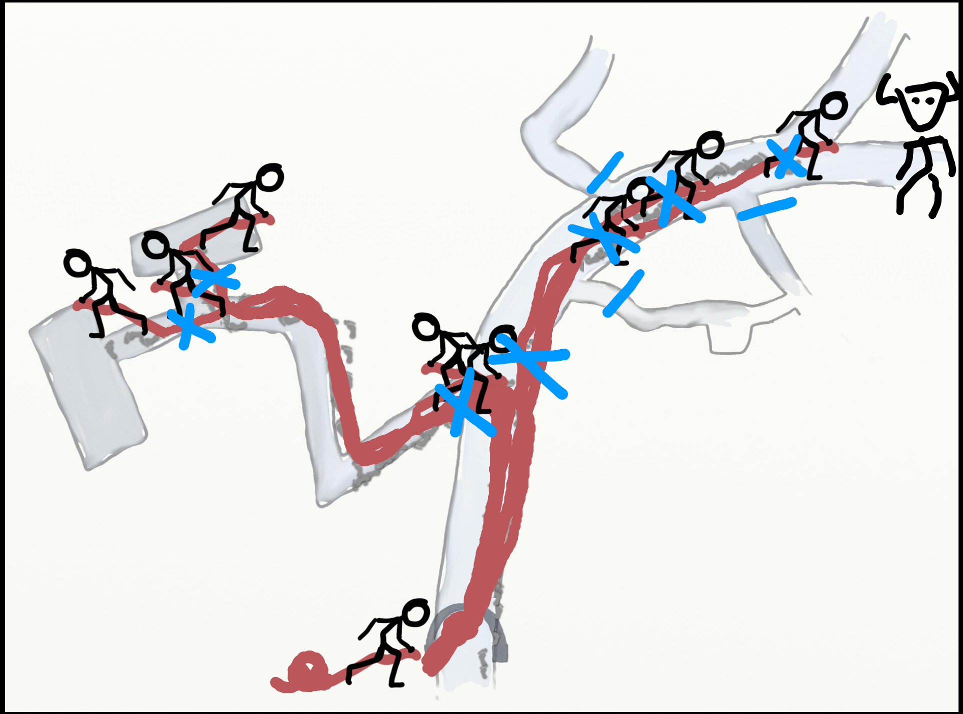
time

```python
# visit all vertices reachable from s
def dfs_recurse(g, s):
    for v in g.vertices:
        v.visited = False
    visit(s)

def visit(v):
    v.visited = True
    for w in v.neighbours:
        if not w.visited:
            visit(w)
```

```python
# visit all vertices reachable from s
def dfs_recurse(g, s):
    for v in g.vertices:
        v.visited = False
    visit(s)

def visit(v):
    v.visited = True
    for w in v.neighbours:
        if not w.visited:
            visit(w)
```

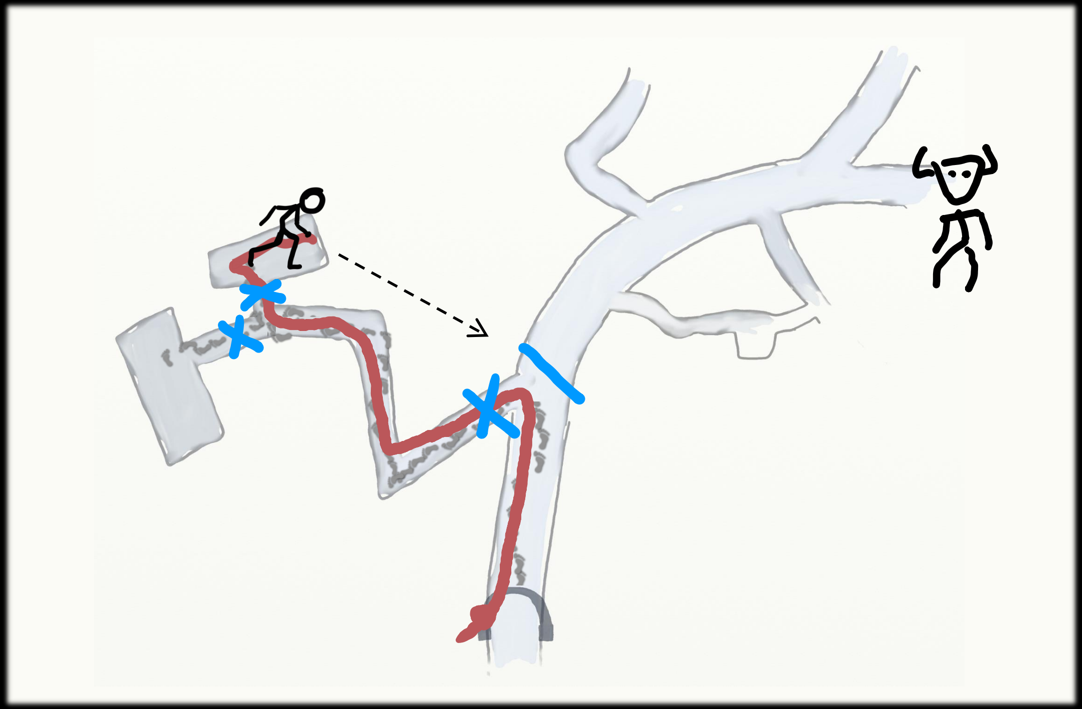
Theseus in the labyrinth of the Minotaur

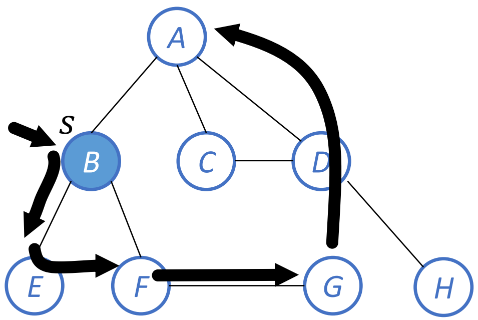
Ariadne gave Theseus a ball of thread. She told him to tie one end at the entrance of the labyrinth, and to unroll the ball as he delved the branching paths. And to mark with chalk those passages he explored. After Theseus slew the Minotaur, he could follow the thread back to the entrance, where Ariadne was waiting.

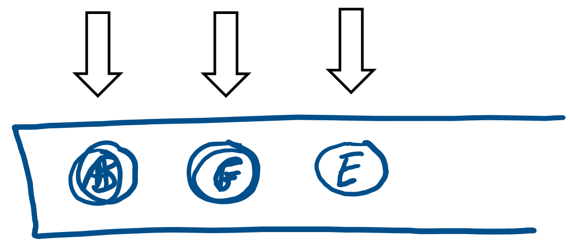


Ariadne's thread

but why not just teleport?








```
# visit all vertices reachable from s
def dfs(g, s):
    for v in g.vertices:
        v.seen = False
    toexplore = Stack([s])
    s.seen = True

    while not toexplore.is_empty():
        v = toexplore.popright()
        for w in v.neighbours:
            if not w.seen:
                toexplore.pushright(w)
                w.seen = True
```

Analysis of running time for stack-based dfs

```python
# visit all vertices reachable from s
def dfs(g, s):
    for v in g.vertices:
        v.seen = False
    toexplore = Stack([s])
    s.seen = True

    while not toexplore.is_empty():
        v = toexplore.popright()
        for w in v.neighbours:
            if not w.seen:
                toexplore.pushright(w)
                w.seen = True
```

Lines 3–4: $O(V)$

Lines 5–6: $O(1)$

Lines 8–9: $O(V)$ since we never visit a vertex more than once

Lines 10–13: $O(E)$ since it looks at every edge from every vertex.

So total cost $O(V+E)$.

Analysis of running time for recursive dfs

```python
# visit all vertices reachable from s
def dfs_recurse(g, s):
    for v in g.vertices:
        v.visited = False
    visit(s)

def visit(v):
    v.visited = True
    for w in v.neighbours:
        if not w.visited:
            visit(w)
```

We Don't try to work out the cost of a single call to visit().

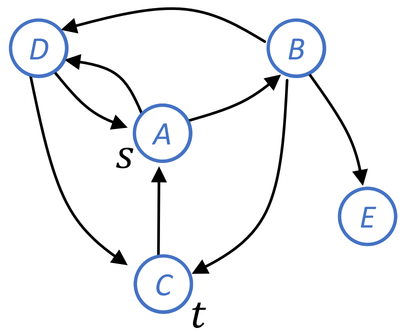
Instead, we just look at aggregate costs.

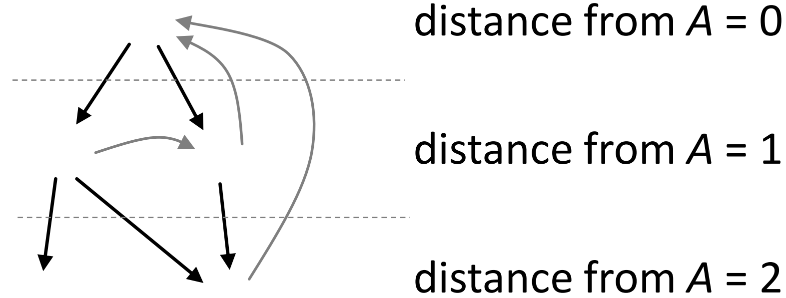
Cost: $O(V+E)$ for the same reason.

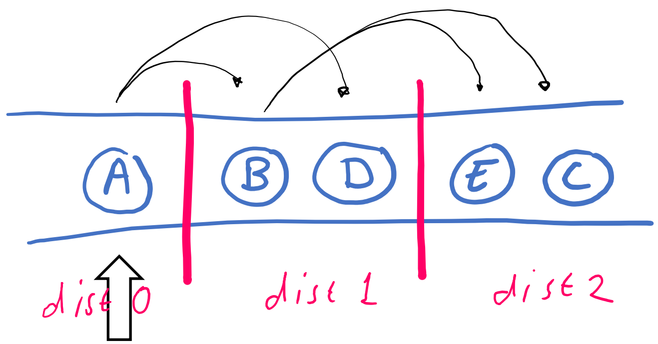
SECTION 5.2

Breadth-first search / finding shortest path

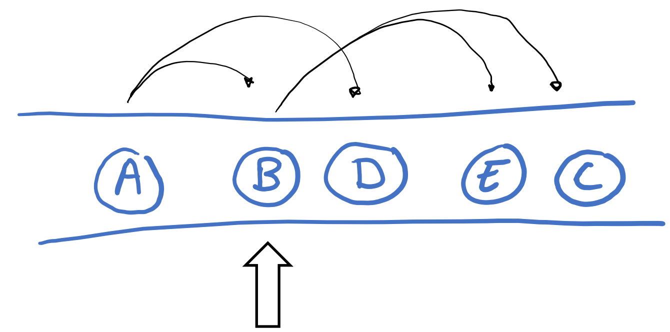
PROBLEM STATEMENT. Given a start vertex s, and an end vertex t, find the shortest path from s to t.







```python
# Visit all the vertices in g reachable from start vertex s
def bfs(g, s):
    for v in g.vertices:
        v.seen = False
    toexplore = Queue([s])
    s.seen = True

    while not toexplore.is_empty():
        v = toexplore.popleft()
        for w in v.neighbours:
            if not w.seen:
                toexplore.pushright(w)
                w.seen = True
```

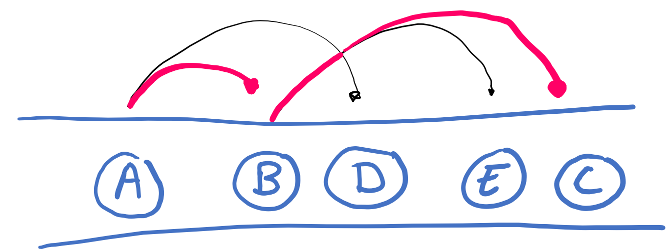


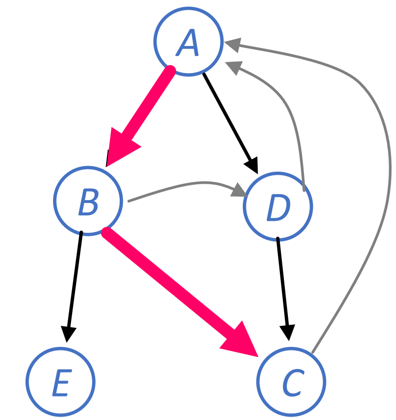
Cost: $O(V+E)$

Same reasoning as for dfs.

```python
# visit all vertices reachable from s
def dfs(g, s):
    for v in g.vertices:
        v.seen = False
    toexplore = Stack([s])
    s.seen = True

    while not toexplore.is_empty():
        v = toexplore.popright()
        for w in v.neighbours:
            if not w.seen:
                toexplore.pushright(w)
                w.seen = True
```

```
# Find a path from s to t, if one exists
def bfs_path(g, s, t):
    for v in g.vertices:
        (v.seen, v.come_from) = (False, None)
...
    while not toexplore.is_empty():
        v = toexplore.popleft()
        for w in v.neighbours:
            if not w.seen:
                toexplore.pushright(w)
                (w.seen, w.come_from) = (True, v)
...
    if t.come_from has not been set:
        there is no path from s to t
    else:
        reconstruct the path from s to t,
        working backwards
```





Q. How might we find a shortest path from *A* to *C*?

3. Algorithm design

Lecture 05 [slides]	3.1 Dynamic programming
Lecture 06 [slides]	Dynamic programming examples
Lecture 07 [slides]	3.2 Greedy algorithms
	First half of example sheet 2 [pdf] **Optional tick: huffman** **Tick 2, due 19 Feb (TBC)**

4. Data structures (intro)

Lecture 08 [slides.pre]	4.1 Memory and pointers 4.1–4.2 List, tree, stack, queue, dictionary 4.7 Hash tables 4.9 Priority queues
	Rest of example sheet 2 [pdf]

5. Graphs and path finding

Lecture 09 [slides.pre]	5, 5.1 Graphs (14:27) 5.2 Depth-first search (11:37) 5.3 Breadth-first search (6:43)
Lecture 10	5.4 Dikstra's algorithm (15:25) plus proof (24:01)
Lecture 11	5.5 Algorithms and proofs (9:29) 5.6 Bellman-Ford (12:13)
Lecture 12	5.7 Dynamic programming (13:06) 5.8 Johnson's algorithm (13:43)
	Example sheet 4 [pdf] **Optional tick: bfs-all from ex4.q6**

6. Graphs and subgraphs

Lecture 13	6.1 Flow networks (9:31) 6.2 Ford-Fulkerson algorithm (21:55)
Lecture 14	6.3 Max-flow min-cut theorem (19:38)
Lecture 15	6.4 Matchings (11:01)

Example sheet 4

Question 6. Modify b ... *rse website, for you to chec* ...

Algorithms tick bfs-all

cl.cam.ac.uk/teaching/2223/Algorithm2/ticks/bfs-all.html

Algorithms tick: bfs-all

Find All Shortest Paths

Breadth-first search can be used to find a shortest path between a pair of vertices. Modify the standard `bfs_path` algorithm so that it returns *all* shortest paths.

Please submit a source file bfs_all.py on Moodle. It should implement a function

```
shortest_paths(g, s, t)

# Find all shortest paths from s to t
# Return a list of paths, each path a list of vertices starting with s and
```

The graph `g` is stored as an adjacency dictionary, for example `g = {0:{1,2}, 1:{}, 2:{1,0}}`. It has a key for every vertex, and the corresponding value is the set of that vertex's neighbours.

EXERCISE: Read the notes / watch the video for section 5.4, to familiarize yourself with Dijkstra's algorithm.

We will spend Friday's lecture going through the <u>proof of correctness</u>.