Ǝloise

∀belard

by Edmund Leighton, 1852—1922

Last time: the InsertSort algorithm, on an array of length n, has running time $\leq {}^1/_2\, k_1 n(n-1) + k_2(n-1)$.

Let's make life easier by only worrying about asymptotic costs.

Definition. Given two functions f and g, both $\mathbb{N} \to \mathbb{R}$, we say $f(n)$ is $O(g(n))$ if

$$\exists \kappa > 0 \text{ and } n_0 \in \mathbb{N} \text{ such that } \forall n \geq n_0, |f(n)| \leq \kappa |g(n)|$$

and we say $f(n)$ is $\Omega(g(n))$ if

$$\exists \delta > 0 \text{ and } n_0 \in \mathbb{N} \text{ such that } \forall n \geq n_0, |f(n)| \geq \delta |g(n)| .$$

If $f(n)$ is $O(g(n))$ and also $\Omega(g(n))$ we say that $f(n)$ is $\Theta(g(n))$.

let $f(n) = \frac{1}{2} k_1 n(n-1) + k_2(n-1)$ k_1, k_2 constants.

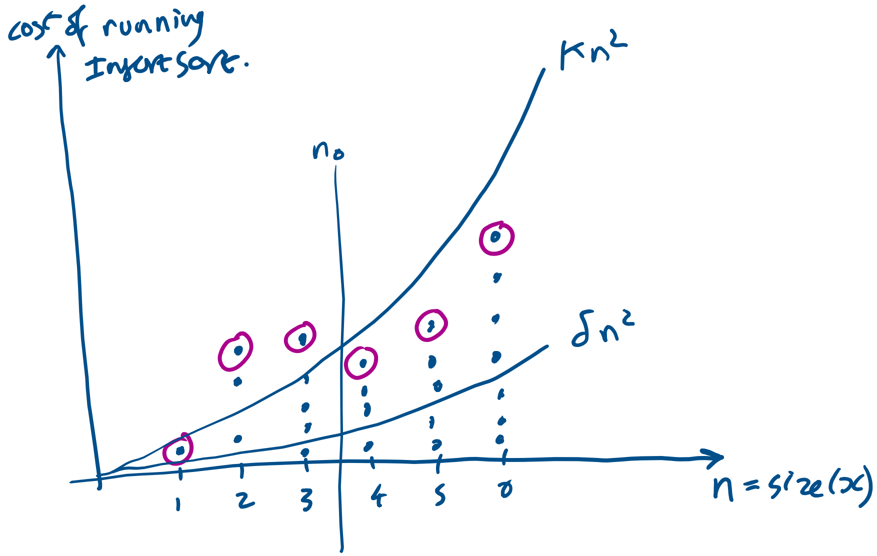
Then $f(n)$ is $O(n^3)$ since $f(n) \leq \frac{1}{2} k_1 n^2 + k_2 n = n^3 \left(\frac{\frac{1}{2}k_1}{n} + \frac{k_2}{n^2} \right) \leq 2n^3$ for $n \geq \max(\frac{1}{2}k_1, k_2)$.

But also $f(n)$ is $O(n^2)$ by similar reasoning. And $O(e^n)$. And ....

Also, $f(n)$ is $\Omega(n^2)$, and $\Omega(\log n)$, and $\Omega(1)$ ..
by similar reasoning.

Since $f(n)$ is $O(n^2)$, and $\Omega(n^2)$, it is $\Theta(n^2)$.

In this course, we're typically interested in an algorithm's worst-case running time as a function of input size.



Plot a dot • for every possible input x.

For each n, circle ⊙ the input that's the worst case.

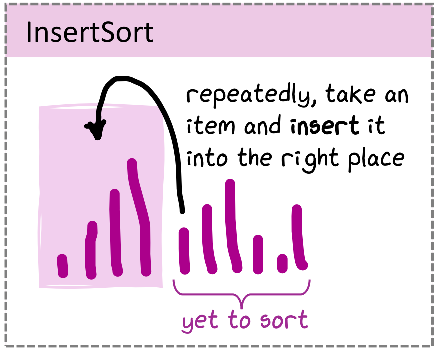
We've shown that for every input x of size n, the cost is $\leq \kappa n^2$ (for some $\kappa > 0$, and sufficiently large n).
In other words, all the blue dots are $\leq \kappa n^2$.

In other words, the purple circles are $\leq \kappa n^2$.
In other words, if we define the worst-case cost to be $h(n) = \max_{x:\, \text{size}(x)=n} \text{cost}(x)$, then $h(n)$ is $O(n^2)$.

Can we find a matching Ω bound, i.e. show that $h(n)$ is $\Omega(n^2)$?
In other words, can we show that the purple circles are $\geq \delta n^2$ (for some $\delta > 0$, and sufficiently large n)?
In other words, can we find for each n a specific input x whose cost is $\geq \delta n^2$?

In this course, we're typically interested in an algorithm's worst-case running time as a function of input size.

```
def insert_sort(x):
  for i in 1..(len(x)-1):
    j = i - 1
    while j >= 0 and x[j] > x[j+1]:
      swap x[j] with x[j+1]
      j = j - 1
```

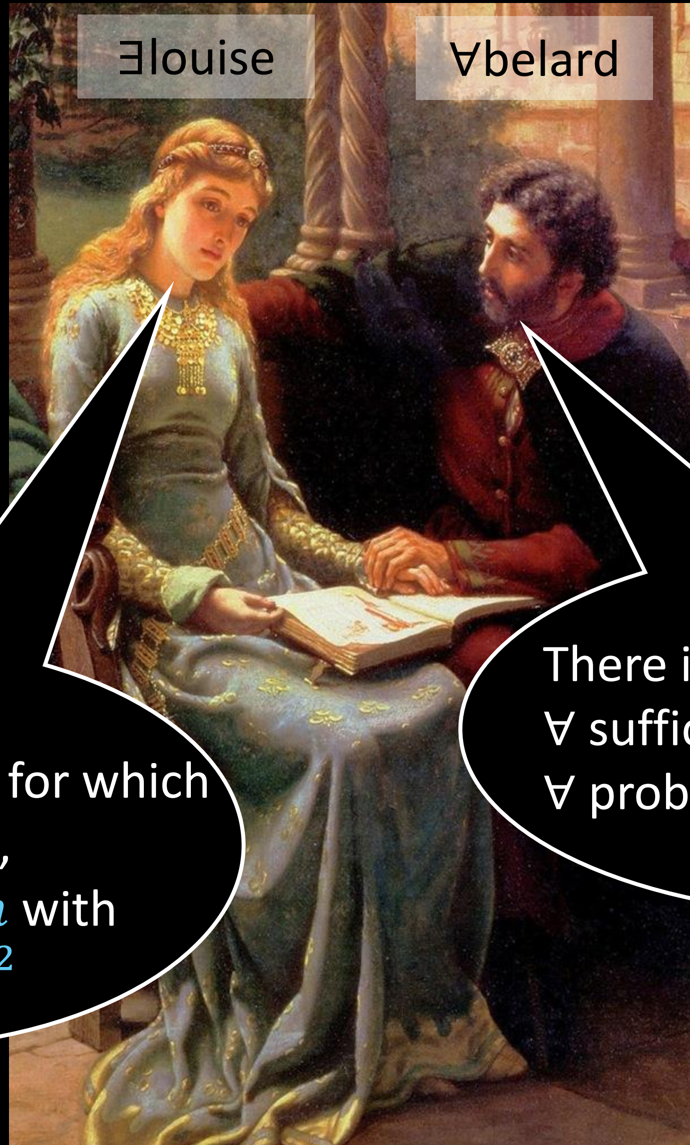


Q. Given an arbitrary n, what is an input of size n that gives the worst possible running time?

For input $[n, n-1, \ldots, 1]$

cost is $\Omega(n^2)$

ChatGPT struggles with ∃ problems.
For example, see the "vulnerability report" at https://hackerone.com/reports/2298307



There is some $\delta > 0$ for which
$\forall$ sufficiently large n,
$\exists$ a problem of size n with
cost $\geq \delta\, n^2$

There is some $\kappa > 0$ for which
$\forall$ sufficiently large n and
$\forall$ problems of size n
cost $\leq \kappa\, n^2$

After we show that our algorithm is $O(n^2)$, it's good manners to also demonstrate that the worst case is $\Omega(n^2)$.

MON Simple sorting algorithms compared

WED Two optimal algorithms

FRI Better than optimal!?

2.5 Minimum cost of sorting

Can we do better than InsertSort's $\Theta(n^2)$ worst-case running time?

Complexity of Comparison Sort?

- typically count the number of comparisons $C(n)$
- there are $n!$ permutations of n elements
- each comparison eliminates *half* of the permutations
  $2^{C(n)} \geq n!$
- therefore $C(n) \geq \log(n!) \approx n \log n - 1.44n$
- The lower bound of comparison is $O(n \log n)$

ALERT! We don't expect to see "lower bound" and "O" in the same sentence!

Properly-stated theorem

Given any sorting alg. A

Let $g_A(x)$ = #comparisons when we run A on input x

Let $f_A(n) = \max_{x : size(x) = n} g_A(x)$

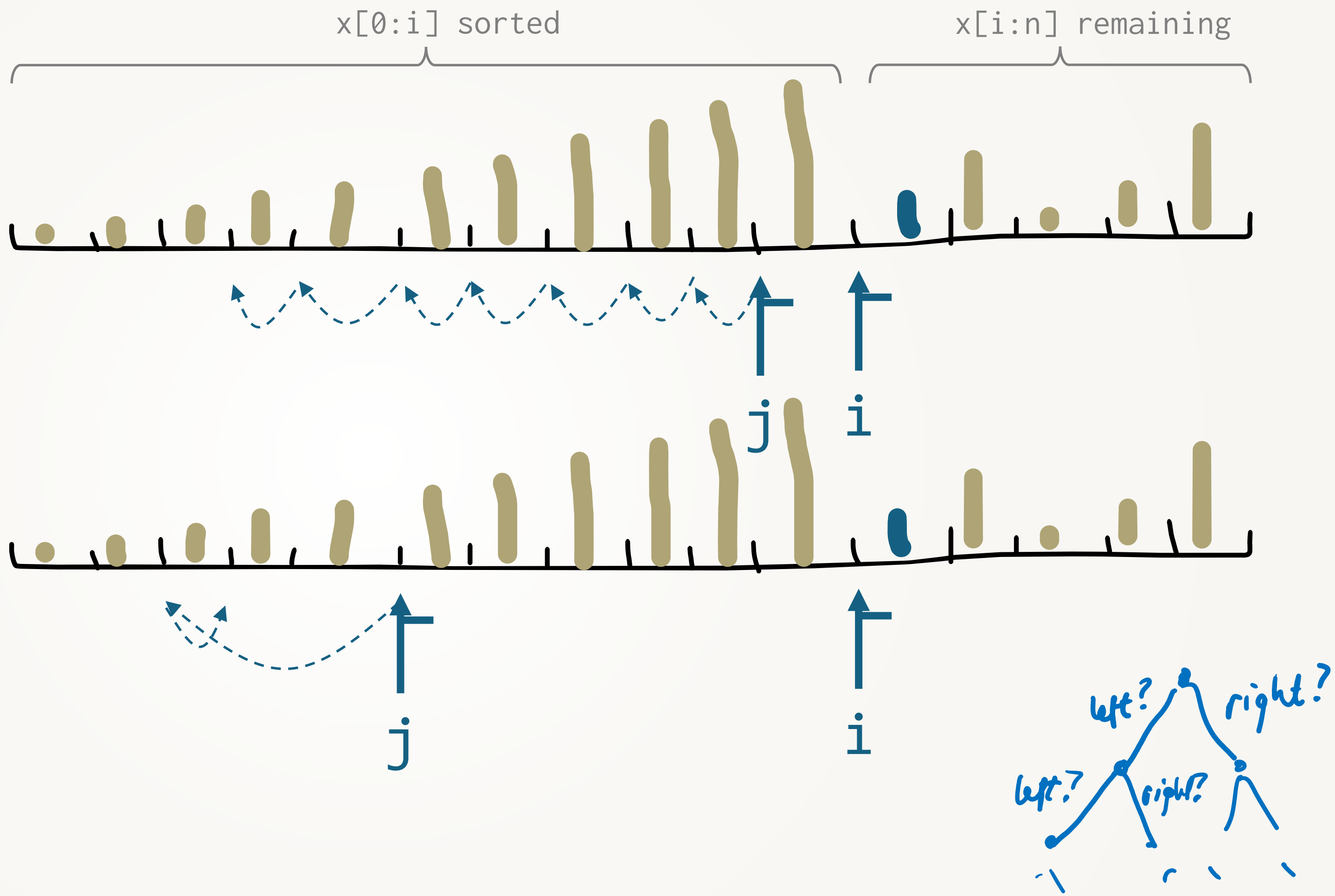
Then $f_A(n)$ is $\Omega(n \log n)$.

§2.7 Binary InsertSort

Can we sort using only $O(n \log n)$ comparisons?

```
def insert_sort(x):
  for i in 1..(len(x)-1):
    do a linear search for
    where x[i] should go, and
    insert it there
```

```
def binary_insert_sort(x):
  for i in 1..(len(x)-1):
    do a **binary** search for
    where x[i] should go, and
    insert it there
```



QUESTION

What's a big-O bound on the number of comparisons for BinaryInsertSort?

$x \le \lceil x \rceil < 1 + x$

$\log n!$ is $\Theta(n \log n)$.

```
def binary_insert_sort(x):
  for i in 1..(len(x)-1):
    do a binary search for
    where x[i] should go, and
    insert it there
```

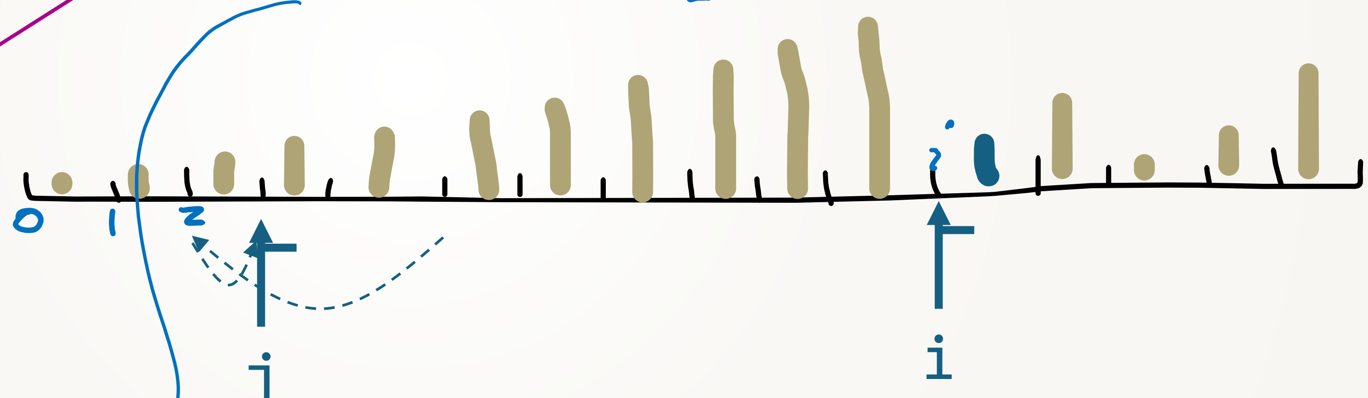
comparisons to place $x[i] \le \lceil \log_2 (i+1) \rceil$

So total # comparisons

$$\le \sum_{i=1}^{n-1} \lceil \log_2 (i+1) \rceil < \sum_{i=1}^{n-1} (1 + \log_2 (i+1))$$

$$= n-1 + \underbrace{\sum_{i=1}^{n-1} \log_2 (i+1)}$$

$$= n-1 + \log_2 [n \times (n-1) \times \dots \times 2] = n-1 + \log_2 n!$$



$$= n-1 + \frac{\log n!}{\log 2} \le n-1 + \frac{\kappa}{\log 2} n \log n$$

for some κ for n suff. large

So total # comparisons is $O(n \log n)$.

We used $\le$ right at the beginning. This "contaminates" all the rest of the working, and means we can only end up with a $O(\cdot)$ conclusion.

QUESTION

What's the asymptotic worst-case number of swaps?

Recall: sum of arithmetic series.

$1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$

```
def binary_insert_sort(x):
  for i in 1..(len(x)-1):
    do a binary search for
    where x[i] should go, and
    insert it there
```

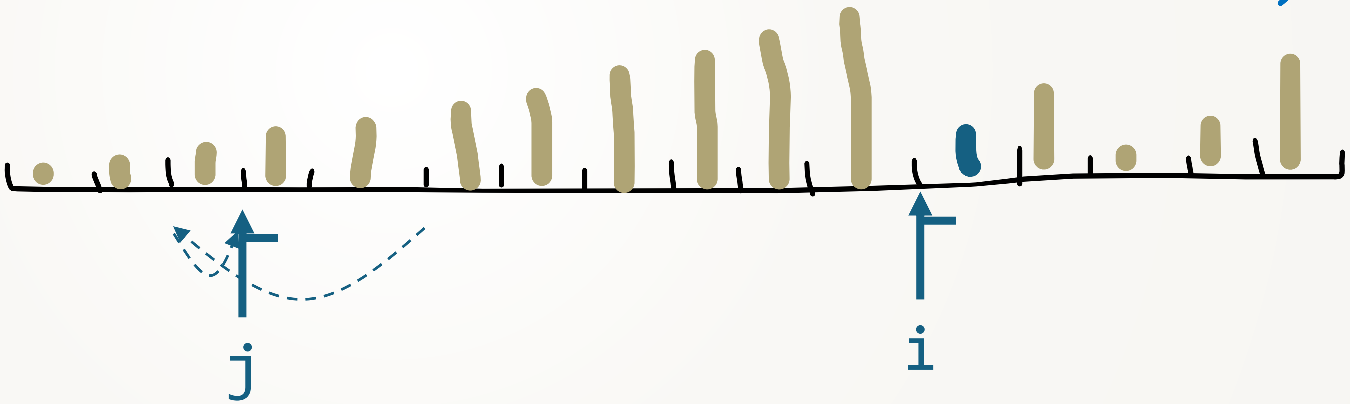
- To place $x[i]$ we might need i swaps.

  Total #swaps $\leq \sum_{i=1}^{n-1} i$

  so worst-case total #swaps is $O(n^2)$

- Thinking of the input $[n, n-1, \ldots, 1]$,

  worst-case total #swaps is $\Omega(n^2)$



§2.6 SelectSort

What's a lower bound for the worst-case number of swaps to sort an array of length n?

Theorem. *For any sorting algorithm, the worst-case number of swaps is* $\Omega(n)$.

Proof. Given arbitrary n, consider the input $x = [2,3,\ldots,n,1]$.

Every item starts in the wrong place, so every item needs to be "touched" by a swap.

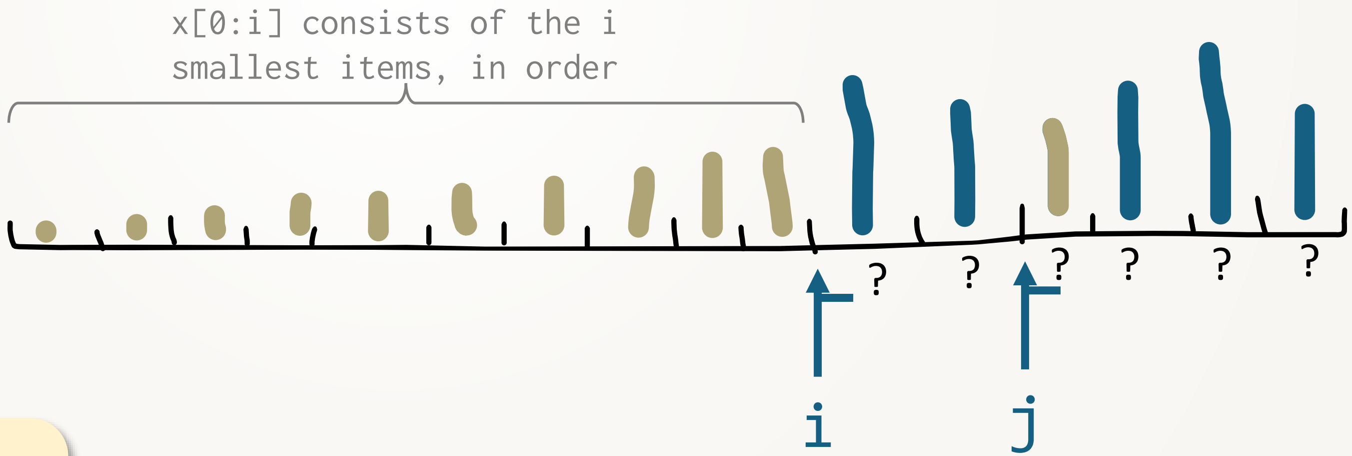
Each swap touches two items.

Thus $\#\text{swaps} \geq \lceil n/2 \rceil$, which is $\Omega(n)$.

Can we sort using only $O(n)$ swaps?

Notation: this means "return the k that achieves the minimum"

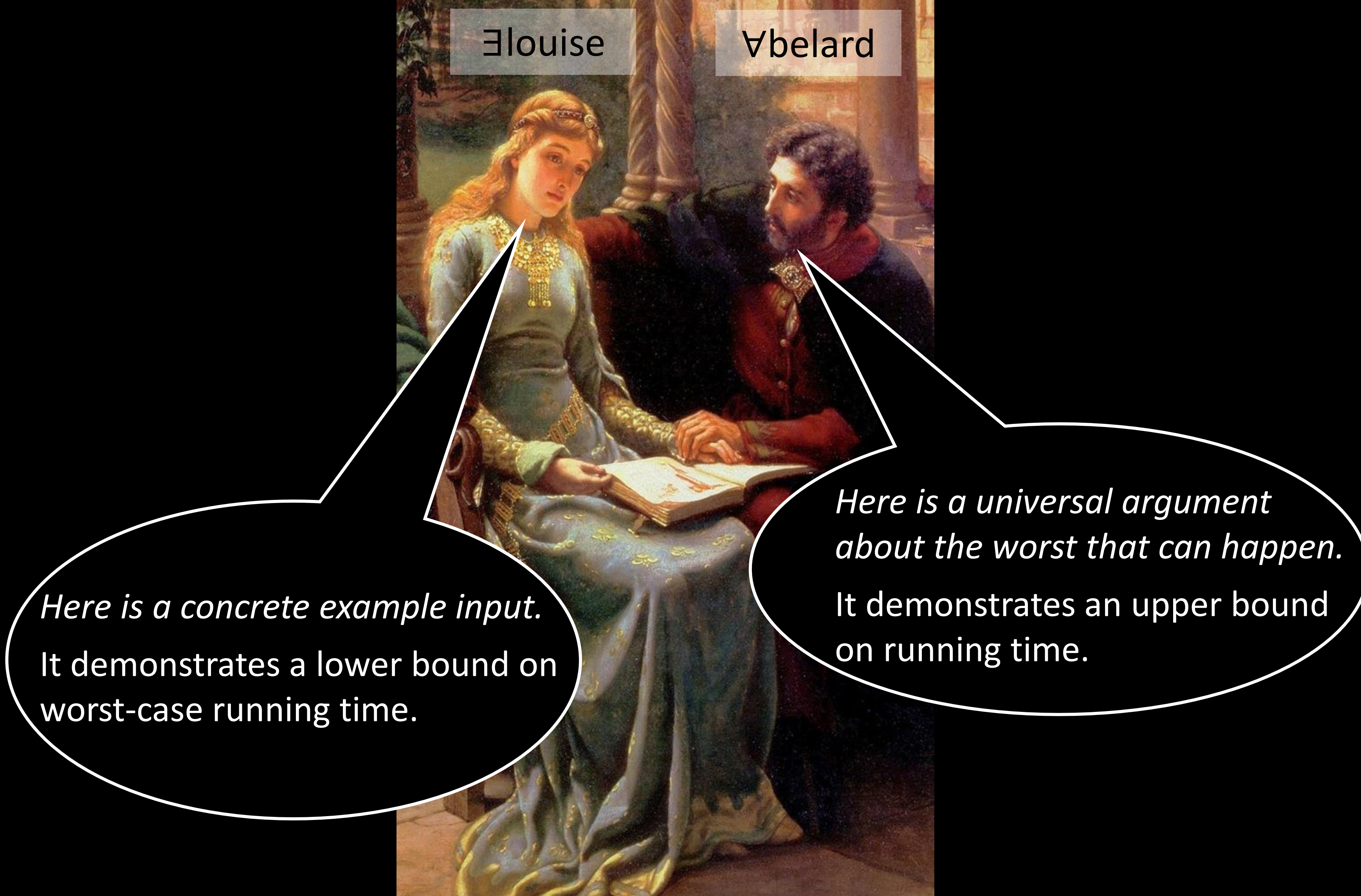
```
def select_sort(x):
  for i in 0..(len(x)-2):
    # Find what belongs in x[i]
    j = arg min x[k]
        i≤k<len(x)
    swap x[i] with x[j]
```



QUESTION
What's the asymptotic worst-case number of comparisons?

Total #comparisons $= \sum_{i=0}^{n-2} (n-i-1) = \Theta(n^2)$.
(for every input)

	comparisons	swaps
any algorithm	worst case is $\Omega(n \log n)$	worst case is $\Omega(n)$
InsertSort	worst case is $O(n^2)$ worst case is $\Omega(n^2)$	worst case is $O(n^2)$ worst case is $\Omega(n^2)$
BinaryInsertSort	worst case is $O(n \log n)$	
SelectSort	every case is $\Theta(n^2)$	worst case is $O(n)$



If our bounds don't agree, we should think harder!

- Can we find a better example, one that hits our upper bound?
- Or maybe the algorithm isn't as bad as we thought: can we find a tighter upper bound?

As well as O and Ω and Θ, we also use o and ω [see notes]

O is pronounced "big-O"
o is pronounced "little-o"
Ω is pronounced "big-Omega"
ω is pronounced "little-omega"

literally means big o!