Discrete Mathematics

Exercises 5

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5. On sets

5.1. Basic exercises

- 1. Prove that \subseteq is a partial order, that is, it is:
 - a) reflexive: \forall sets $A. A \subseteq A$
 - b) transitive: \forall sets A, B, C. $(A \subseteq B \land B \subseteq C) \Longrightarrow A \subseteq C$
 - c) antisymmetric: \forall sets A, B. $(A \subseteq B \land B \subseteq A) \iff A = B$
- 2. Prove the following statements:
 - a) \forall sets A. $\emptyset \subseteq A$
 - b) $\forall \text{ sets } A. \ (\forall x. x \notin A) \iff A = \emptyset$
- 3. Find the union, and intersection of:
 - a) $\{1, 2, 3, 4, 5\}$ and $\{-1, 1, 3, 5, 7\}$
 - b) $\{x \in \mathbb{R} \mid x > 7\}$ and $\{x \in \mathbb{N} \mid x > 5\}$
- 4. Find the Cartesian product and disjoint union of $\{1, 2, 3, 4, 5\}$ and $\{-1, 1, 3, 5, 7\}$.
- 5. Let $I = \{2, 3, 4, 5\}$ and for each $i \in I$, let $A_i = \{i, i + 1, i 1, 2 \cdot i\}$.
 - a) List the elements of all sets A_i for $i \in I$.
 - b) Let $\{A_i \mid i \in I\}$ stand for $\{A_2, A_3, A_4, A_5\}$. Find $\bigcup \{A_i \mid i \in I\}$ and $\bigcap \{A_i \mid i \in I\}$.
- 6. Let U be a set. For all $A, B \in \mathcal{P}(U)$, prove that:
 - a) $A^{c} = B \iff (A \cup B = U \land A \cap B = \emptyset)$
 - b) Double complement elimination: $(A^c)^c = A$
 - c) The de Morgan laws: $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$

5.2. Core exercises

1. Prove that for all for all sets U and subsets $A, B \subseteq U$:

a) $\forall X. A \subseteq X \land B \subseteq X \iff (A \cup B) \subseteq X$ b) $\forall Y. Y \subseteq A \land Y \subseteq B \iff Y \subseteq (A \cap B)$

- 2. Either prove or disprove that, for all sets A and B,
 - a) $A \subseteq B \Longrightarrow \mathcal{P}(A) \subseteq \mathcal{P}(B)$
 - b) $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$
 - c) $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$

- d) $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$
- e) $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$
- 3. Let U be a set. For all $A, B \in \mathcal{P}(U)$ prove that the following statements are equivalent.

a) $A \cup B = B$ b) $A \subseteq B$ c) $A \cap B = A$ d) $B^c \subseteq A^c$

4. For sets *A*, *B*, *C*, *D*, prove or disprove at least three of the following statements:

a)
$$(A \subseteq C \land B \subseteq D) \Longrightarrow A \times B \subseteq C \times D$$

- b) $(A \cup C) \times (B \cup D) \subseteq (A \times B) \cup (C \times D)$
- c) $(A \times C) \cup (B \times D) \subseteq (A \cup B) \times (C \cup D)$
- d) $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$
- e) $(A \times B) \cup (A \times D) \subseteq A \times (B \cup D)$
- 5. For sets *A*, *B*, *C*, *D*, prove or disprove at least three of the following statements:
 - a) $(A \subseteq C \land B \subseteq D) \Longrightarrow A \uplus B \subseteq C \uplus D$
 - b) $(A \cup B) \uplus C \subseteq (A \uplus C) \cup (B \uplus C)$
 - c) $(A \uplus C) \cup (B \uplus C) \subseteq (A \cup B) \uplus C$
 - d) $(A \cap B) \uplus C \subseteq (A \uplus C) \cap (B \uplus C)$
 - e) $(A \uplus C) \cap (B \uplus C) \subseteq (A \cap B) \uplus C$
- 6. Prove the following properties of the big unions and intersections of a family of sets $\mathcal{F} \subseteq \mathcal{P}(A)$:

a)
$$\forall U \subseteq A. \ (\forall X \in \mathcal{F}. X \subseteq U) \iff \bigcup \mathcal{F} \subseteq U$$

b)
$$\forall L \subseteq A. \ (\forall X \in \mathcal{F}. \ L \subseteq X) \iff L \subseteq \bigcap \mathcal{F}$$

- 7. Let *A* be a set.
 - a) For a family $\mathcal{F} \subseteq \mathcal{P}(A)$, let $\mathcal{U} \triangleq \{ U \subseteq A \mid \forall S \in \mathcal{F} . S \subseteq U \}$. Prove that $\bigcup \mathcal{F} = \bigcap \mathcal{U}$.
 - b) Analogously, define the family $\mathcal{L} \subseteq \mathcal{P}(A)$ such that $\bigcap \mathcal{F} = \bigcup \mathcal{L}$. Also prove this statement.

5.3. Optional advanced exercises

1. Prove that for all families of sets \mathcal{F}_1 and \mathcal{F}_2 ,

$$(\bigcup \mathcal{F}_1) \cup (\bigcup \mathcal{F}_2) = \bigcup (\mathcal{F}_1 \cup \mathcal{F}_2)$$

State and prove the analogous property for intersections of non-empty families of sets.

2. For a set U, prove that $(\mathcal{P}(U), \subseteq, \cup, \cap, U, \emptyset, (\cdot)^c)$ is a Boolean algebra.