Discrete Mathematics

Exercises 6

Marcelo Fiore Ohad Kammar Dima Szamozvancev

6. On relations

6.1. Basic exercises

1. Let $A = \{1, 2, 3, 4\}, B = \{a, b, c, d\}$ and $C = \{x, y, z\}$. Let $R = \{(1, a), (2, d), (3, a), (3, b), (3, d)\}: A \rightarrow B$ and $S = \{(b, x), (b, y), (c, y), (d, z)\}: B \rightarrow C$.

Draw the internal diagrams of the relations. What is the composition $S \circ R: A \rightarrow C$?

- 2. Prove that relational composition is associative and has the identity relation as the neutral element.
- 3. For a relation $R: A \rightarrow B$, let its opposite, or dual relation, $R^{op}: B \rightarrow A$ be defined by:

$$b R^{\mathrm{op}} a \iff a R b$$

For $R, S: A \rightarrow B$ and $T: B \rightarrow C$, prove that:

- a) $R \subseteq S \Longrightarrow R^{\mathrm{op}} \subseteq S^{\mathrm{op}}$
- b) $(R \cap S)^{\text{op}} = R^{\text{op}} \cap S^{\text{op}}$
- c) $(R \cup S)^{\text{op}} = R^{\text{op}} \cup S^{\text{op}}$
- d) $(T \circ S)^{\mathrm{op}} = S^{\mathrm{op}} \circ T^{\mathrm{op}}$

6.2. Core exercises

- 1. Let $R, R' \subseteq A \times B$ and $S, S' \subseteq B \times C$ be two pairs of relations and assume $R \subseteq R'$ and $S \subseteq S'$. Prove that $S \circ R \subseteq S' \circ R'$.
- 2. Let $\mathcal{F} \subseteq \mathcal{P}(A \times B)$ and $\mathcal{G} \subseteq \mathcal{P}(B \times C)$ be two collections of relations from A to B and from B to C, respectively. Prove that

$$\left(\bigcup \mathcal{G}\right) \circ \left(\bigcup \mathcal{F}\right) = \bigcup \{S \circ R \mid R \in \mathcal{F}, S \in \mathcal{G}\} : A \leftrightarrow C$$

Recall that the notation $\{S \circ R : A \leftrightarrow C \mid R \in \mathcal{F}, S \in \mathcal{G}\}$ is common syntactic sugar for the formal definition $\{T \in \mathcal{P}(A \times C) \mid \exists R \in \mathcal{F}, \exists S \in \mathcal{G}, T = S \circ R\}$. Hence,

$$T \in \{S \circ R \in A \leftrightarrow C \mid R \in \mathcal{F}, S \in \mathcal{G}\} \iff \exists R \in \mathcal{F}, \exists S \in \mathcal{G}, T = S \circ R\}$$

What happens in the case of big intersections?

- 3. Suppose *R* is a relation on a set *A*. Prove that
 - a) *R* is reflexive iff $id_A \subseteq R$
 - b) *R* is symmetric iff $R = R^{op}$

- c) *R* is transitive iff $R \circ R \subseteq R$
- d) *R* is antisymmetric iff $R \cap R^{op} \subseteq id_A$
- 4. Let *R* be an arbitrary relation on a set *A*, for example, representing an undirected graph. We are interested in constructing the smallest transitive relation (graph) containing *R*, called the *transitive closure* of *R*.
 - a) We define the family of relations which are transitive supersets of *R*:

$$\mathcal{T}_R \triangleq \{Q: A \rightarrow A \mid R \subseteq Q \text{ and } Q \text{ is transitive } \}$$

R is not necessarily going to be an element of this family, as it might not be transitive. However, *R* is a *lower bound* for T_R , as it is a subset of every element of the family.

Prove that the set $\bigcap T_R$ is the transitive closure for *R*.

b) $\bigcap \mathcal{T}_R$ is the intersection of an infinite number of relations so it's difficult to compute the transitive closure this way. A better approach is to start with R, and keep adding the missing connections until we get a transitive graph. This can be done by repeatedly composing R with itself: after n compositions, all paths of length n in the graph represented by R will have a transitive connection between their endpoints.

Prove that the (at least once) iterated composition $R^{\circ+} \triangleq R \circ R^{\circ*}$ is the transitive closure for R, i.e. it coincides with the greatest lower bound of $\mathcal{T}_R: R^{\circ+} = \bigcap \mathcal{T}_R$. Hint: show that $R^{\circ+}$ is both an element and a lower bound of \mathcal{T}_R .

7. On partial functions

7.1. Basic exercises

- 1. Let $A_2 = \{1, 2\}$ and $A_3 = \{a, b, c\}$. List the elements of the sets $PFun(A_i, A_j)$ for $i, j \in \{2, 3\}$. *Hint*: there may be quite a few, so you can think of ways of characterising all of them without giving an explicit listing.
- 2. Prove that a relation $R: A \rightarrow B$ is a partial function iff $R \circ R^{op} \subseteq id_B$.
- 3. Prove that the identity relation is a partial function, and that the composition of partial functions is a partial function.

7.2. Core exercises

- 1. Show that $(PFun(A, B), \subseteq)$ is a partial order. What is its least element, if it exists?
- 2. Let $\mathcal{F} \subseteq PFun(A, B)$ be a non-empty collection of partial functions from A to B.
 - a) Show that $\bigcap \mathcal{F}$ is a partial function.
 - b) Show that $\bigcup \mathcal{F}$ need not be a partial function by defining two partial functions $f, g: A \rightarrow B$ such that $f \cup g: A \rightarrow B$ is a non-functional relation.
 - c) Let $h: A \rightarrow B$ be a partial function. Show that if every element of \mathcal{F} is below h then $\bigcup \mathcal{F}$ is a partial function.

8. On functions

8.1. Basic exercises

- 1. Let $A_2 = \{1, 2\}$ and $A_3 = \{a, b, c\}$. List the elements of the sets Fun (A_i, A_j) for $i, j \in \{2, 3\}$.
- 2. Prove that the identity partial function is a function, and the composition of functions yields a function.
- 3. Prove or disprove that $(Fun(A, B), \subseteq)$ is a partial order.
- 4. Find endofunctions $f, g: A \rightarrow A$ such that $f \circ g \neq g \circ f$.

8.2. Core exercises

- 1. A relation $R: A \rightarrow B$ is said to be *total* if $\forall a \in A$. $\exists b \in B$. a R b. Prove that this is equivalent to $id_A \subseteq R^{op} \circ R$. Conclude that a relation $R: A \rightarrow B$ is a function iff $R \circ R^{op} \subseteq id_B$ and $id_A \subseteq R^{op} \circ R$.
- 2. Let $\chi : \mathcal{P}(U) \to (U \Rightarrow [2])$ be the function mapping subsets $S \subseteq U$ to their characteristic functions $\chi_S : U \to [2]$.
 - a) Prove that for all $x \in U$,
 - $\chi_{A\cup B}(x) = (\chi_A(x) \lor \chi_B(x)) = \max(\chi_A(x), \chi_B(x))$
 - $\chi_{A\cap B}(x) = (\chi_A(x) \land \chi_B(x)) = \min(\chi_A(x), \chi_B(x))$
 - $\chi_{A^c}(x) = \neg(\chi_A(x)) = (1 \chi_A(x))$
 - b) For what construction A?B on sets A and B does it hold that

$$\chi_{A?B}(x) = (\chi_A(x) \oplus \chi_B(x)) = (\chi_A(x) +_2 \chi_B(x))$$

for all $x \in U$, where \oplus is the *exclusive or* operator? Prove your claim.

8.3. Optional advanced exercise

Consider a set A together with an element $a \in A$ and an endofunction $f : A \rightarrow A$.

Say that a relation $R: \mathbb{N} \leftrightarrow A$ is (a, f)-closed whenever

R(0,a) and $\forall n \in \mathbb{N}, x \in A. R(n,x) \Longrightarrow R(n+1,f(x))$

Define the relation $F : \mathbb{N} \rightarrow A$ as

$$F \triangleq \bigcap \{R: \mathbb{N} \to A \mid R \text{ is } (a, f) \text{-closed} \}$$

- a) Prove that F is (a, f)-closed.
- b) Prove that F is total, that is: $\forall n \in \mathbb{N}$. $\exists y \in A$. F(n, y).
- c) Prove that *F* is a function $\mathbb{N} \to A$, that is: $\forall n \in \mathbb{N}$. $\exists ! y \in A$. F(n, y).

Hint: Proceed by induction. Observe that, in view of the previous item, to show that $\exists ! y \in A$. F(k, y) it suffices to exhibit an (a, f)-closed relation R_k such that $\exists ! y \in A$. $R_k(k, y)$. (Why?) For instance, as the relation $R_0 = \{(m, y) \in \mathbb{N} \times A \mid m = 0 \Longrightarrow y = a\}$ is (a, f)-closed one has that $F(0, y) \Longrightarrow R_0(0, y) \Longrightarrow y = a$.

d) Show that if h is a function $\mathbb{N} \to A$ with h(0) = a and $\forall n \in \mathbb{N}$. h(n+1) = f(h(n)) then h = F.

Thus, for every set A together with an element $a \in A$ and an endofunction $f : A \to A$ there exists a unique function $F : \mathbb{N} \to A$, typically said to be *inductively defined*, satisfying the recurrence relation

$$F(n) = \begin{cases} a & \text{for } n = 0\\ f(F(n-1)) & \text{for } n \ge 1 \end{cases}$$