Randomised Algorithms

Lecture 9: Approximation Algorithms: MAX-3-CNF and Vertex-Cover

Thomas Sauerwald (tms41@cam.ac.uk)

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Outline

Randomised Approximation

MAX-3-CNF

Weighted Vertex Cover

Approximation Ratio for Randomised Approximation Algorithms

Approximation Ratio

A randomised algorithm for a problem has approximation ratio $\rho(n)$, if for any input of size n, the expected cost (value) $\mathbf{E}[\,C\,]$ of the returned solution and optimal cost C^* satisfy:

$$\max\left(\frac{\mathbf{E}[\,C\,]}{C^*}, \frac{C^*}{\mathbf{E}[\,C\,]}\right) \leq \rho(n).$$

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- Maximisation problem: $\frac{C^*}{\mathbf{E}[C]} \geq 1$
- Minimisation problem: $\frac{\mathbf{E}[C]}{C^*} \geq 1$

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Randomised Approximation Schemes

An approximation scheme is an approximation algorithm, which given any input and $\epsilon > 0$, is a $(1+\epsilon)$-approximation algorithm.

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- It is a polynomial-time approximation scheme (PTAS) if for any fixed $\epsilon > 0$, the runtime is polynomial in n. For example, $O(n^{2/\epsilon})$.
- It is a fully polynomial-time approximation scheme (FPTAS) if the runtime is polynomial in both $1/\epsilon$ and n. For example, $O((1/\epsilon)^2 \cdot n^3)$.

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Relaxation of the satisfiability problem. Want to compute how “close” the formula to being satisfiable is.

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$$(x_1 \vee x_3 \vee \overline{x_4}) \wedge (x_1 \vee \overline{x_3} \vee \overline{x_5}) \wedge (x_2 \vee \overline{x_4} \vee x_5) \wedge (\overline{x_1} \vee x_2 \vee \overline{x_3})$$

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Idea: What about assigning each variable uniformly and independently at random?

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Given an instance of MAX-3-CNF with n variables $x_1, x_2, \ldots, x_n$ and m clauses, the randomised algorithm that sets each variable independently at random is a randomised $8/7$-approximation algorithm.

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Follows from the previous Corollary.

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$$\mathbf{E}[Y] = \frac{1}{2} \cdot \mathbf{E}[Y \mid x_1 = 1] + \frac{1}{2} \cdot \mathbf{E}[Y \mid x_1 = 0].$$

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Algorithm: Assign x_1 so that the conditional expectation is maximised and recurse.

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Given an instance of MAX-3-CNF with n variables $x_1, x_2, \ldots, x_n$ and m clauses, the randomised algorithm that sets each variable independently at random is a polynomial-time randomised 8/7-approximation algorithm.

One could prove that the probability to satisfy $(7/8) \cdot m$ clauses is at least $1/(8m)$

$$\mathbf{E}[\,Y\,] = \frac{1}{2} \cdot \mathbf{E}[\,Y \mid x_1 = 1\,] + \frac{1}{2} \cdot \mathbf{E}[\,Y \mid x_1 = 0\,].$$

Y is defined as in the previous proof.

One of the two conditional expectations is at least $\mathbf{E}[\,Y\,]$

GREEDY-3-CNF(ϕ, n, m)

for $j = 1, 2, \ldots, n$

Compute $\mathbf{E}[\,Y \mid x_1 = v_1 \ldots, x_{j-1} = v_{j-1}, x_j = 1\,]$

Compute $\mathbf{E}[\,Y \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = 0\,]$

Let $x_j = v_j$ so that the conditional expectation is maximised

return the assignment $v_1, v_2, \ldots, v_n$

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GREEDY-3-CNF(ϕ, n, m)

```
for j = 1, 2, ..., n
    Compute E[ Y | x_1 = v_1 ..., x_{j-1} = v_{j-1}, x_j = 1 ]
    Compute E[ Y | x_1 = v_1, ..., x_{j-1} = v_{j-1}, x_j = 0 ]
    Let x_j = v_j so that the conditional expectation is maximised
return the assignment v_1, v_2, ..., v_n
```

Skip Analysis

Analysis of GREEDY-3-CNF(ϕ, n, m)

Theorem

GREEDY-3-CNF(ϕ, n, m) is a polynomial-time $8/7$-approximation.

Analysis of GREEDY-3-CNF(ϕ, n, m)

This algorithm is deterministic.

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computable in $O(1)$

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Analysis of Greedy-3-CNF(ϕ, n, m)

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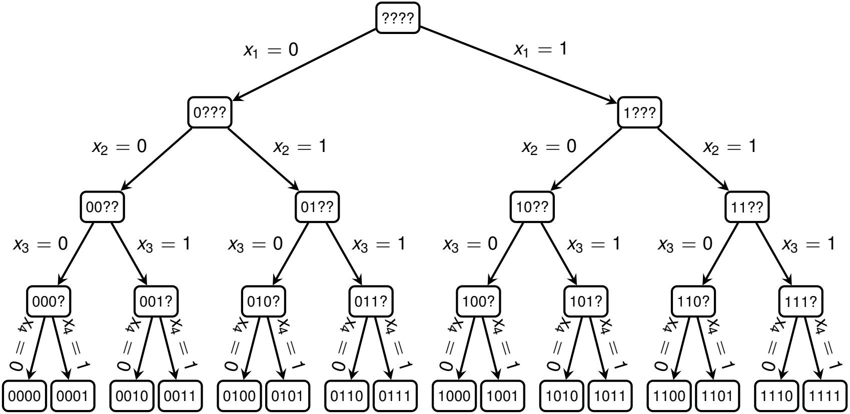
Go to Conclusion

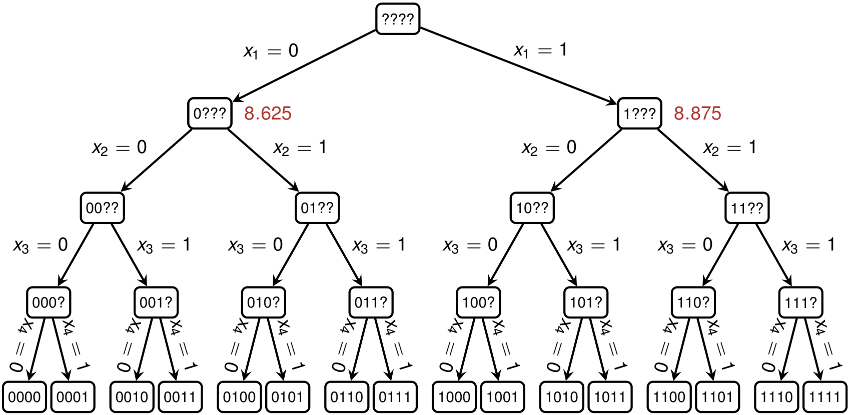
Run of GREEDY-3-CNF(φ, n, m)

$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_4}) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_3} \vee x_4) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee x_3 \vee x_4) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$



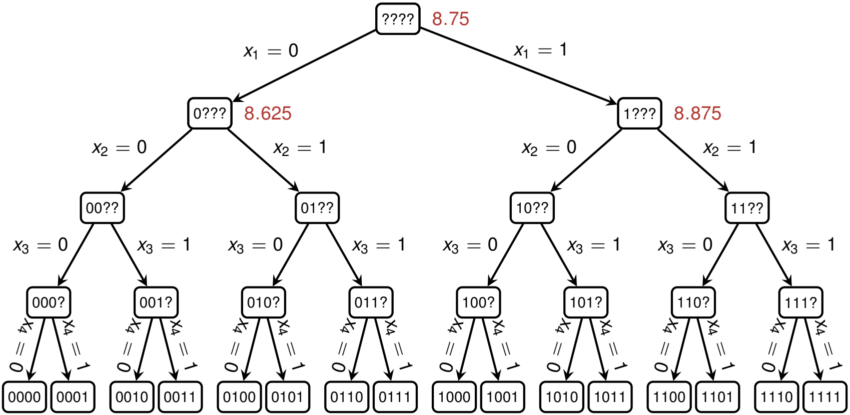
Run of GREEDY-3-CNF(φ, n, m)

$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_4}) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_3} \vee x_4) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee x_3 \vee x_4) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$



Run of GREEDY-3-CNF(φ, n, m)

$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_4}) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_3} \vee x_4) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee x_3 \vee x_4) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$$



Run of GREEDY-3-CNF(φ, n, m)

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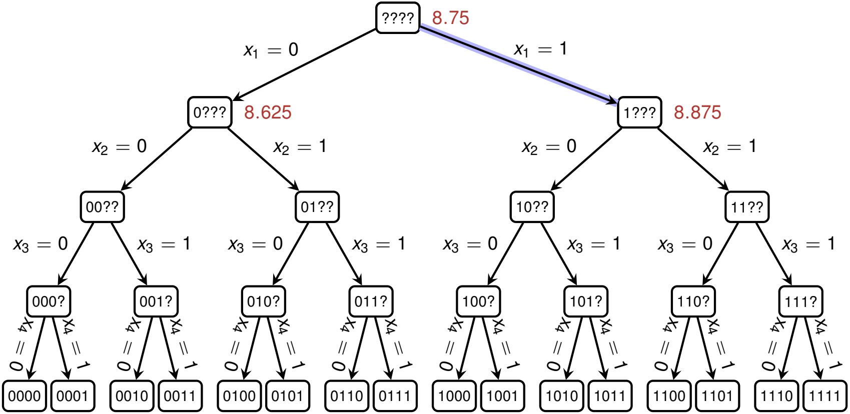

Run of GREEDY-3-CNF(φ, n, m)

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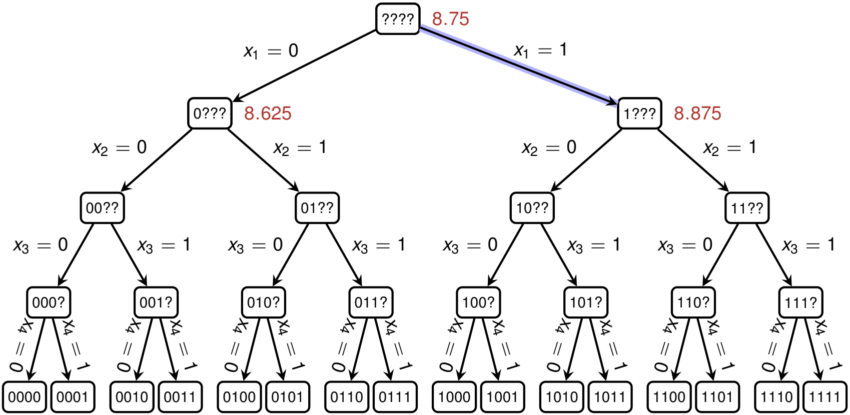
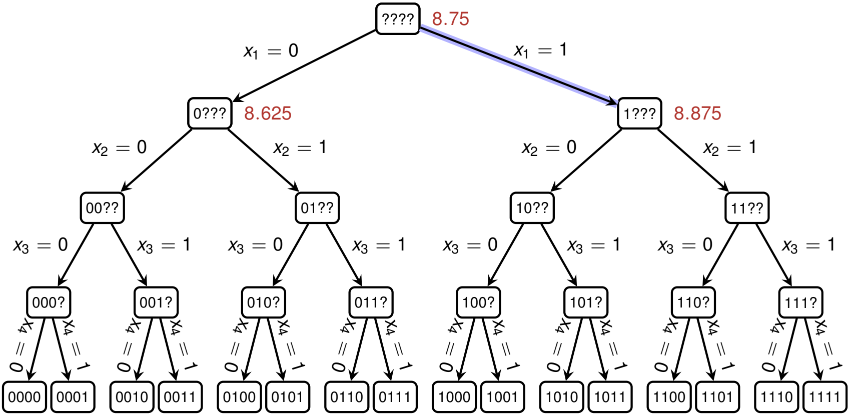


Run of GREEDY-3-CNF(φ, n, m)

$$1 \wedge 1 \wedge 1 \wedge (\overline{x_3} \vee x_4) \wedge 1 \wedge (\overline{x_2} \vee \overline{x_3}) \wedge (x_2 \vee x_3) \wedge (\overline{x_2} \vee x_3) \wedge 1 \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$$



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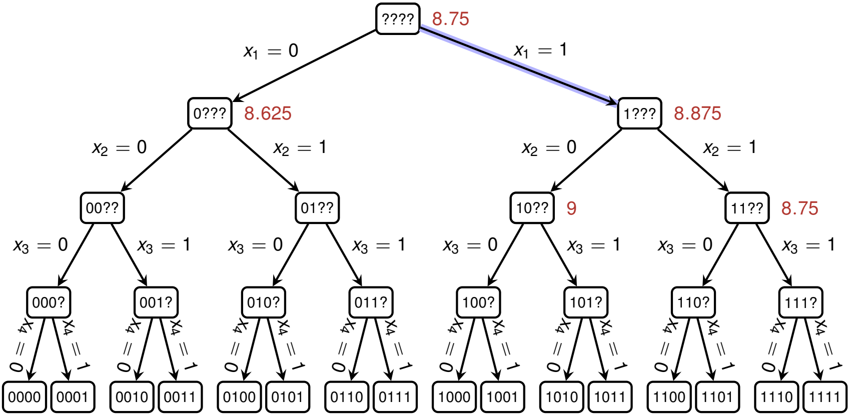

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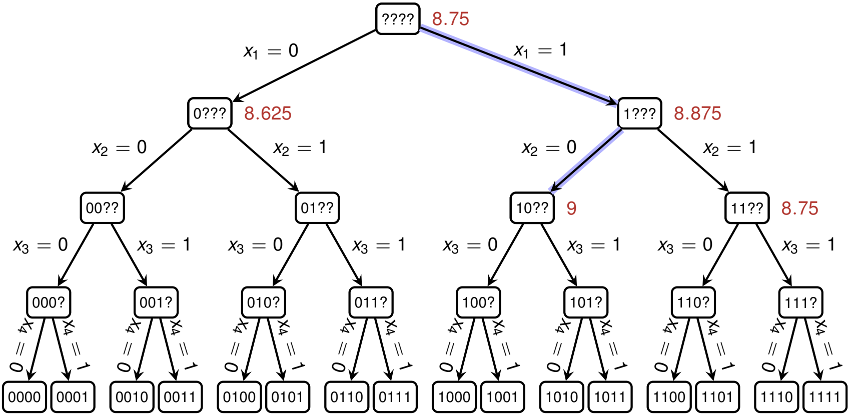

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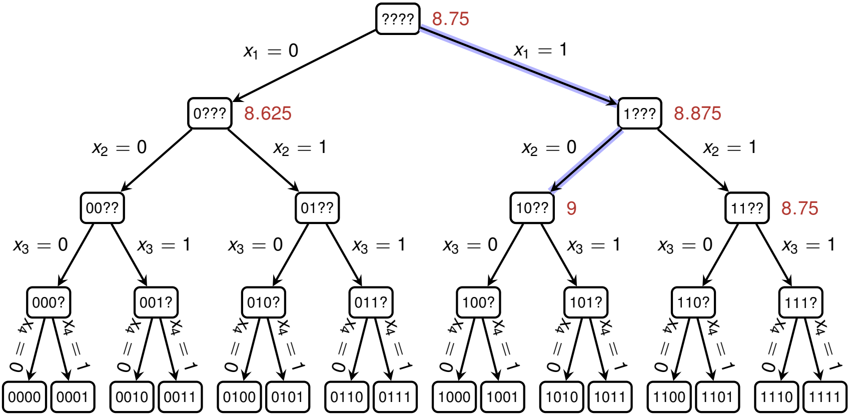

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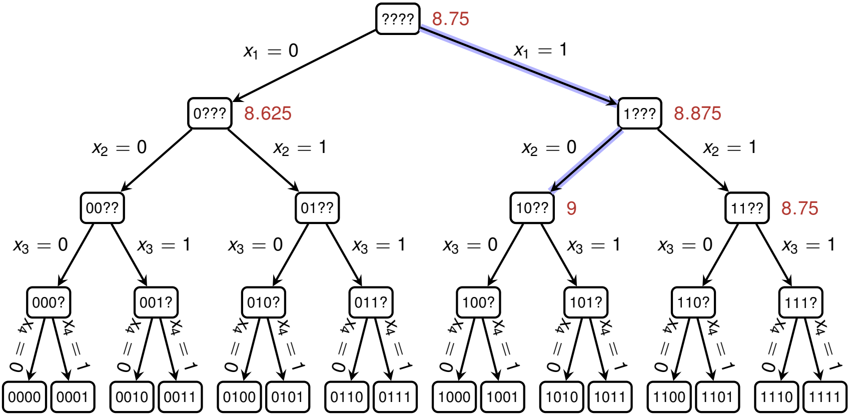

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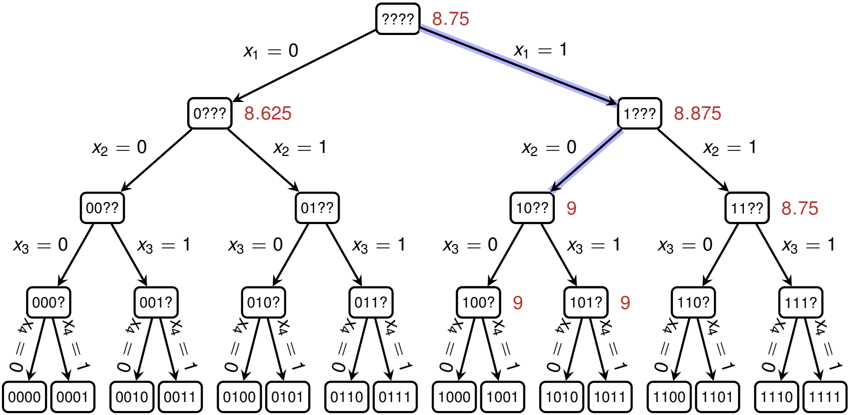

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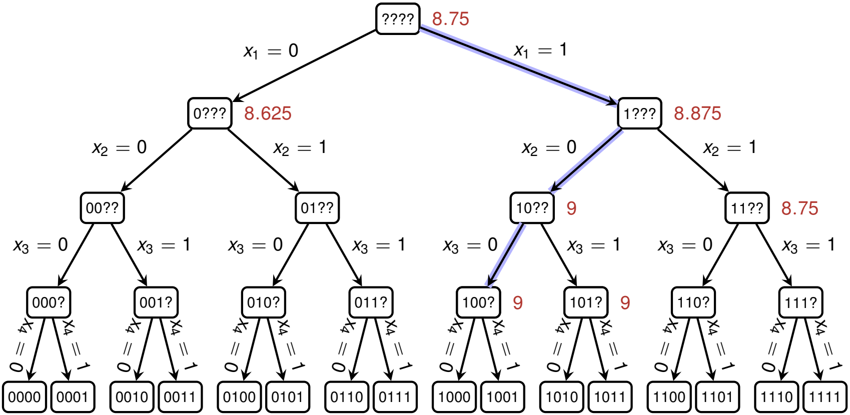

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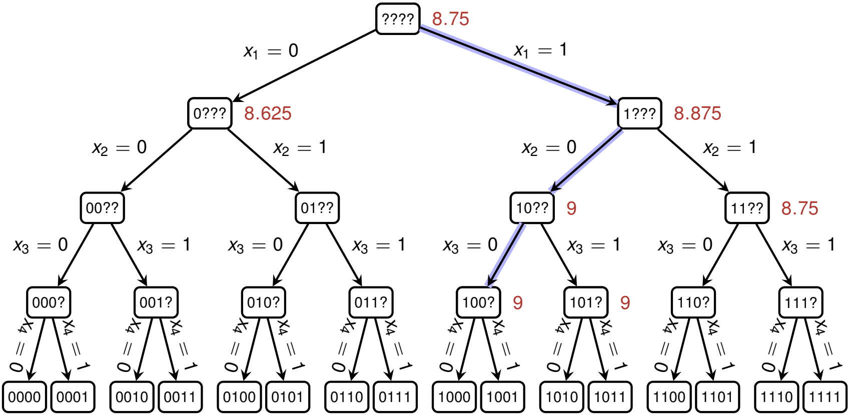
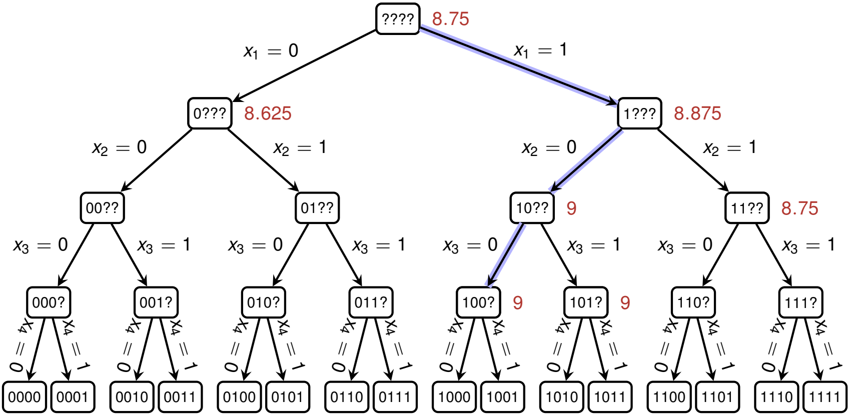


Run of GREEDY-3-CNF(φ, n, m)

$1 \wedge 1 \wedge 1 \wedge 1 \wedge 1 \wedge 1 \wedge 0 \wedge 1 \wedge 1 \wedge 1$



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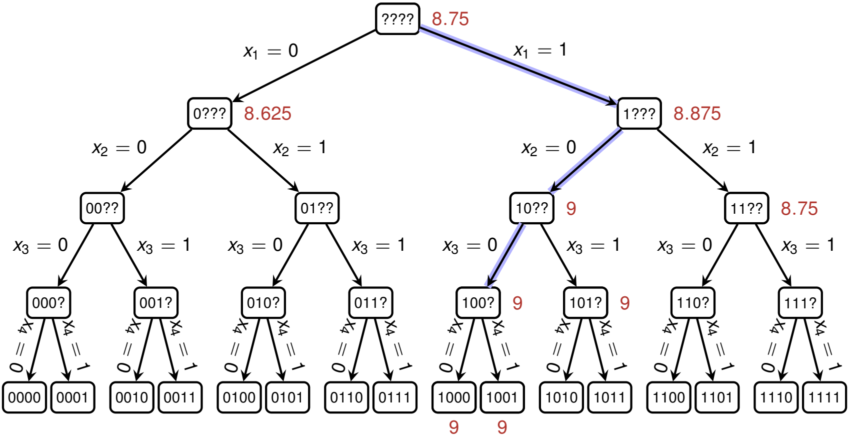

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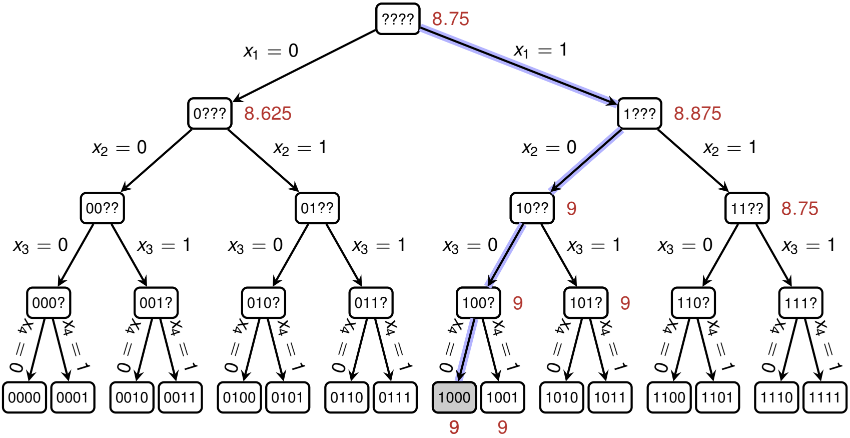

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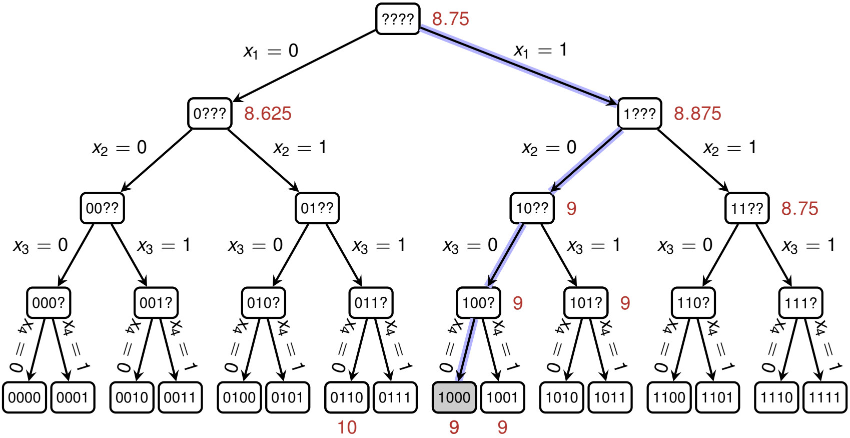

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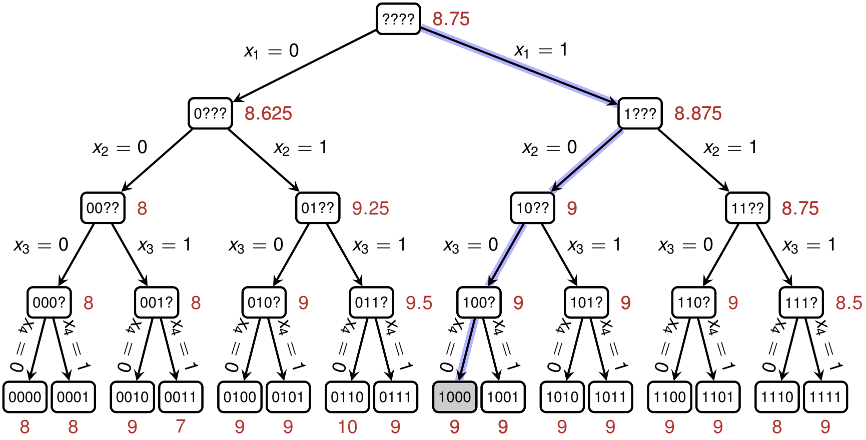

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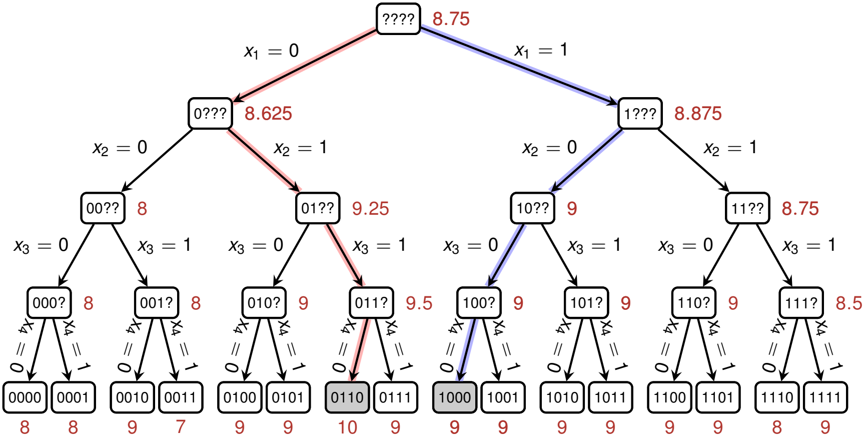

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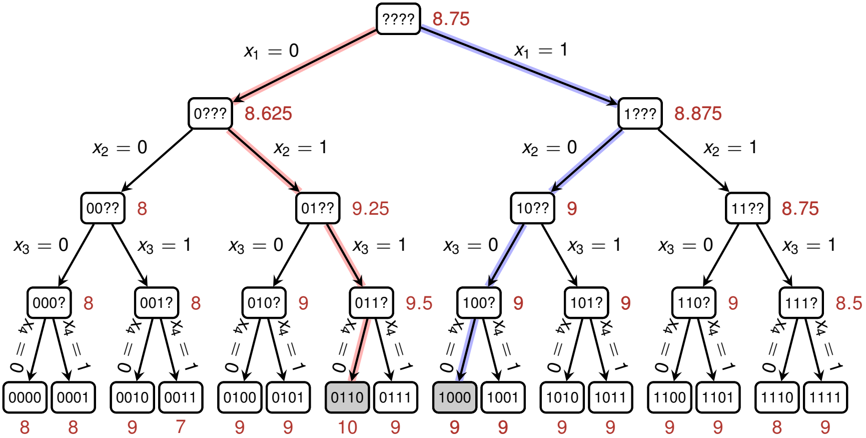

Returned solution satisfies 9 out of 10 clauses, but the formula is satisfiable.

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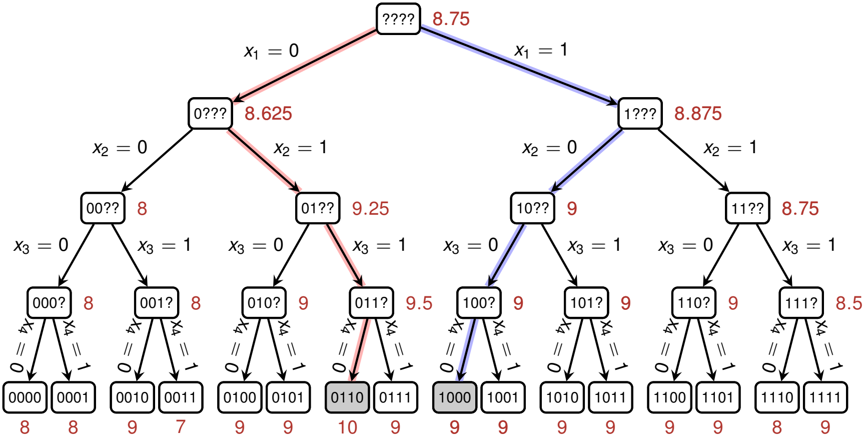

Go to Analysis

MAX-3-CNF: Concluding Remarks

Theorem 35.6

Given an instance of MAX-3-CNF with n variables $x_1, x_2, \ldots, x_n$ and m clauses, the randomised algorithm that sets each variable independently at random is a randomised $8/7$-approximation algorithm.

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GREEDY-3-CNF(ϕ, n, m) is a polynomial-time $8/7$-approximation.

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Theorem (Hastad'97)

For any $\epsilon > 0$, there is no polynomial time $8/7 - \epsilon$ approximation algorithm of MAX3-CNF unless P=NP.

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For any $\epsilon > 0$, there is no polynomial time $8/7 - \epsilon$ approximation algorithm of MAX3-CNF unless P=NP.

Essentially there is nothing smarter than just guessing!

Source of Image: Stefan Szeider, TU Vienna

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Outline

Randomised Approximation

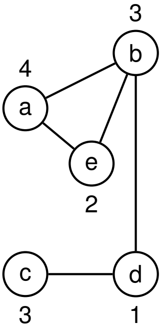
MAX-3-CNF

Weighted Vertex Cover

The Weighted Vertex-Cover Problem

Vertex Cover Problem

- Given: Undirected, vertex-weighted graph $G = (V, E)$
- Goal: Find a minimum-weight subset $V' \subseteq V$ such that if $(u, v) \in E(G)$, then $u \in V'$ or $v \in V'$.



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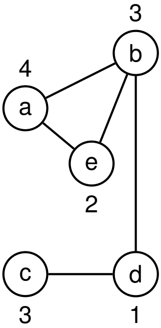

Question: How can we deal with graphs that have **negative** weights?

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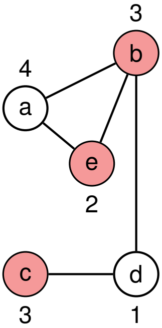

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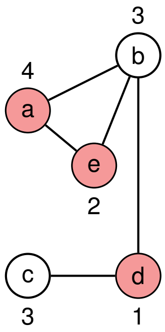

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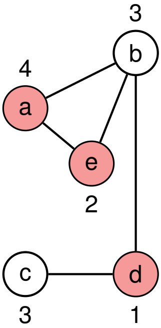

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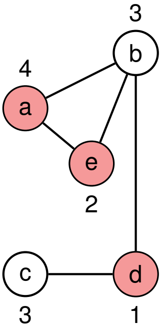

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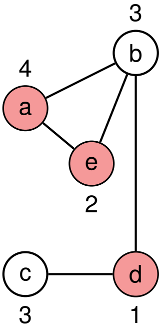

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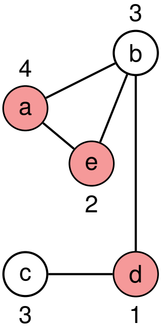

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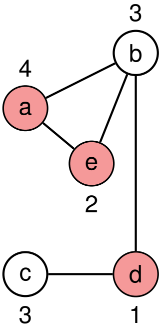

Applications:

- Every edge forms a task, and every vertex represents a person/machine which can execute that task
- Weight of a vertex could be salary of a person
- Perform all tasks with the minimal amount of resources

A Greedy Approach working for Unweighted Vertex Cover

APPROX-VERTEX-COVER(G)

$C = \emptyset$
$E' = G.E$
while $E' \neq \emptyset$
    let (u, v) be an arbitrary edge of E'
    $C = C \cup \{u, v\}$
    remove from E' every edge incident on either u or v
return C

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This algorithm is a 2-approximation for **unweighted graphs**!

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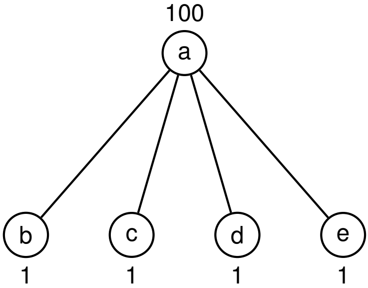

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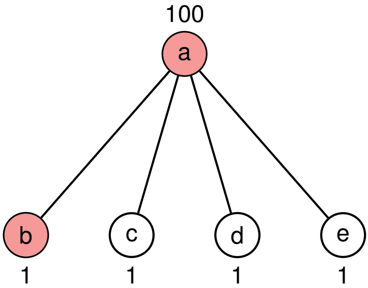

Computed solution has weight 101

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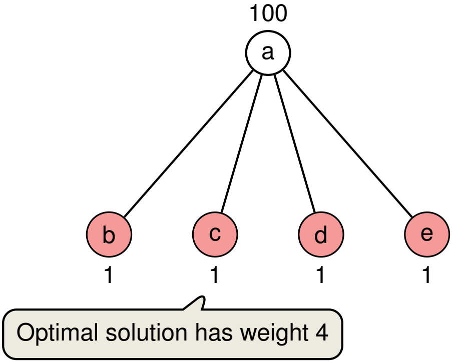

Invoking an (Integer) Linear Program

Idea: Round the solution of an associated linear program.

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0-1 Integer Program

$$
\begin{array}{llll}
\text{minimize} & \displaystyle\sum_{v \in V} w(v)x(v) & & \\
\text{subject to} & x(u) + x(v) \geq 1 & & \text{for each } (u,v) \in E \\
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optimum is a lower bound on the optimal weight of a minimum weight-cover.

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Rounding Rule: if $x(v) \geq 1/2$ then round up, otherwise round down.

The Algorithm

APPROX-MIN-WEIGHT-VC(G, w)

$C = \emptyset$
compute $\bar{x}$, an optimal solution to the linear program
for each $v \in V$
    if $\bar{x}(v) \geq 1/2$
        $C = C \cup \{v\}$
return C

The Algorithm

```
APPROX-MIN-WEIGHT-VC(G, w)
C = ∅
compute x̄, an optimal solution to the linear program
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Theorem 35.7

APPROX-MIN-WEIGHT-VC is a polynomial-time 2-approximation algorithm for the minimum-weight vertex-cover problem.

The Algorithm

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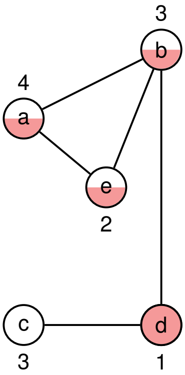
Theorem 35.7

APPROX-MIN-WEIGHT-VC is a polynomial-time 2-approximation algorithm for the minimum-weight vertex-cover problem.

is polynomial-time because we can solve the linear program in polynomial time

Example of APPROX-MIN-WEIGHT-VC

$\overline{x}(a) = \overline{x}(b) = \overline{x}(e) = \frac{1}{2}, \overline{x}(d) = 1, \overline{x}(c) = 0$



fractional solution of LP
with weight $= 5.5$

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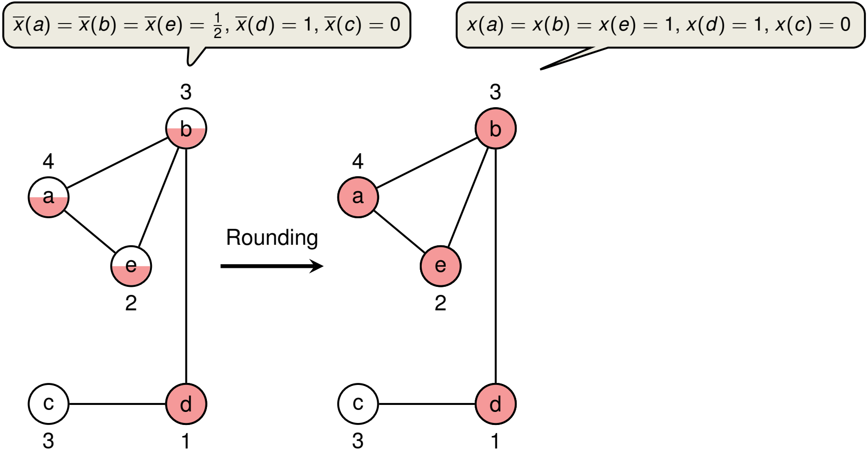

fractional solution of LP
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rounded solution of LP
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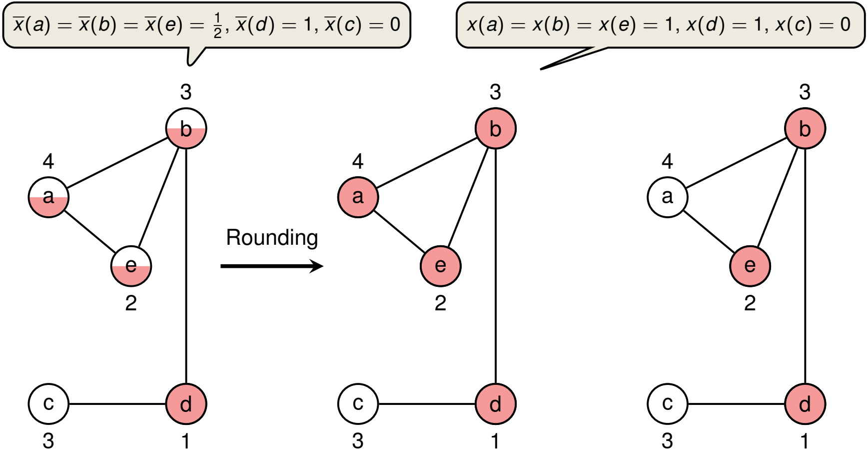

fractional solution of LP with weight $= 5.5$

rounded solution of LP with weight $= 10$

optimal solution with weight $= 6$

Approximation Ratio

Proof (Approximation Ratio is 2 and Correctness):

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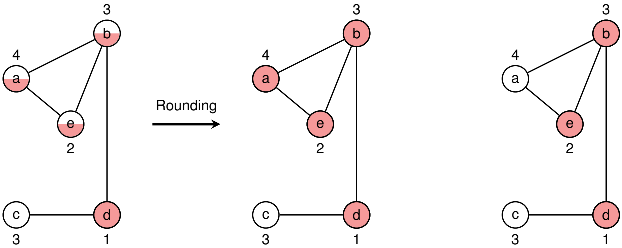

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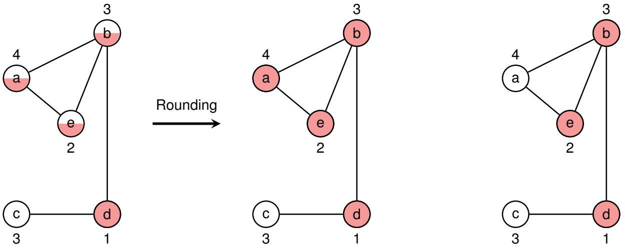
Proof (Approximation Ratio is 2 and Correctness):

- Let C^* be an optimal solution to the minimum-weight vertex cover problem



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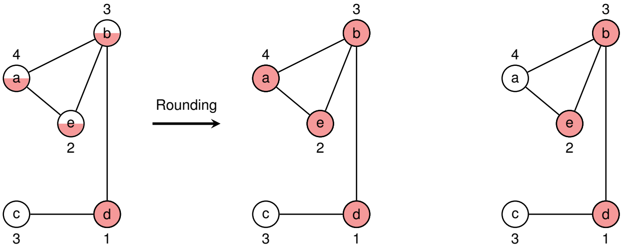

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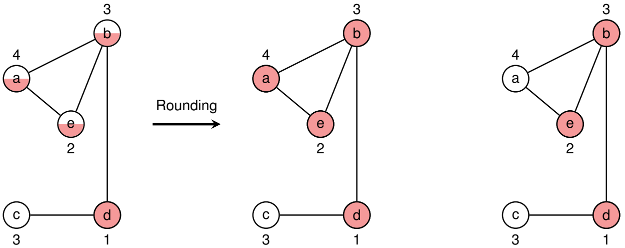

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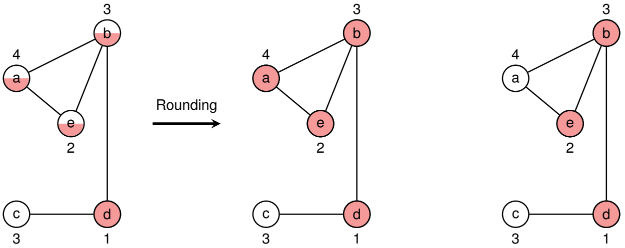

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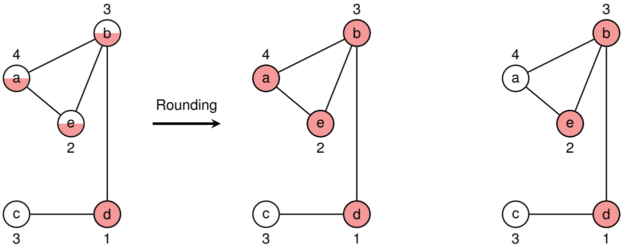

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Proof (Approximation Ratio is 2 and Correctness):

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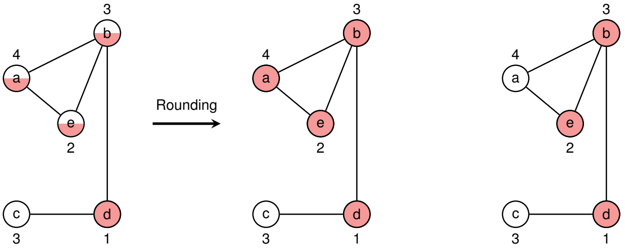

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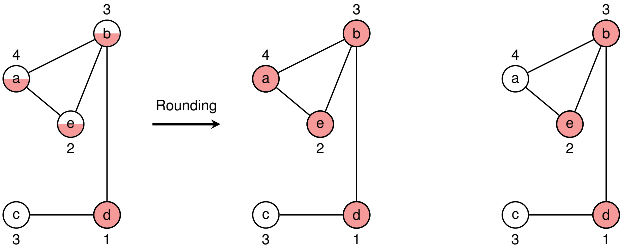

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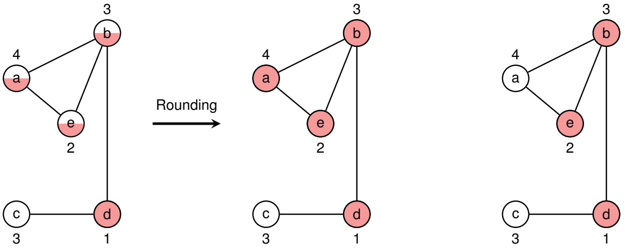

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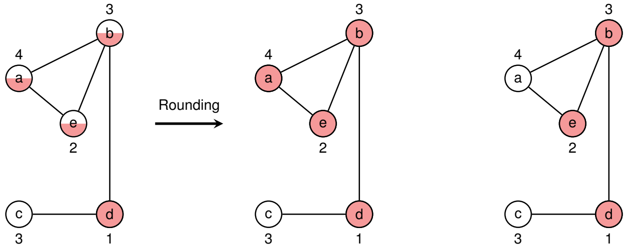

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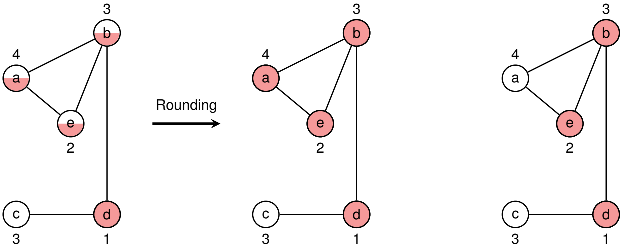

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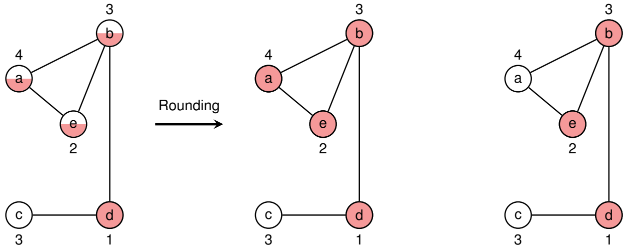

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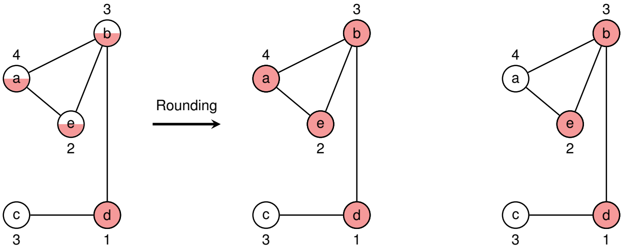

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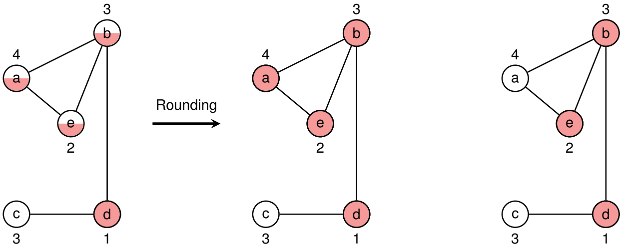

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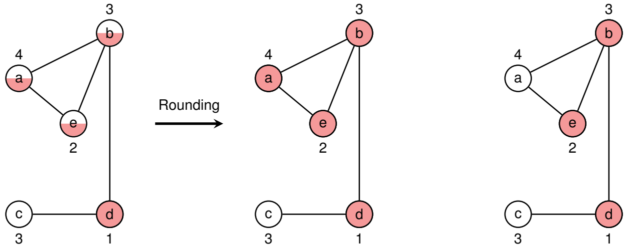

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