Randomised Algorithms

Lecture 6: Linear Programming: Introduction

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Lent 2023

UNIVERSITY OF
CAMBRIDGE

Outline

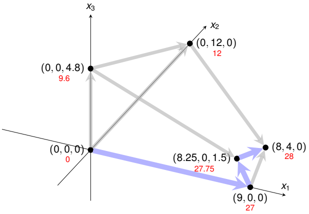
Introduction

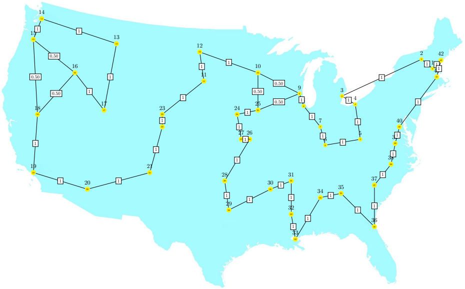
A Simple Example of a Linear Program

Formulating Problems as Linear Programs

Standard and Slack Forms

Introduction





- linear programming is a powerful tool in optimisation
- inspired more sophisticated techniques such as quadratic optimisation, convex optimisation, integer programming and semi-definite programming
- we will later use the connection between linear and integer programming to tackle several problems (Vertex-Cover, Set-Cover, TSP, satisfiability)

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What are Linear Programs?

Linear Programming (informal definition)

- maximise or minimise an objective, given limited resources (competing constraint)
- constraints are specified as (in)equalities
- objective function and constraints are linear

A Simple Example of a Linear Optimisation Problem

- Laptop

A Simple Example of a Linear Optimisation Problem

- Laptop
    - selling price to retailer: 1,000 GBP

A Simple Example of a Linear Optimisation Problem

- Laptop
    - selling price to retailer: 1,000 GBP
    - glass: 4 units

A Simple Example of a Linear Optimisation Problem

- Laptop
    - selling price to retailer: 1,000 GBP
    - glass: 4 units
    - copper: 2 units

A Simple Example of a Linear Optimisation Problem

- Laptop
  - selling price to retailer: 1,000 GBP
  - glass: 4 units
  - copper: 2 units
  - rare-earth elements: 1 unit

A Simple Example of a Linear Optimisation Problem

- Laptop
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- Smartphone

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- You have a daily supply of:

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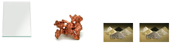

- Smartphone
  - selling price to retailer: 1,000 GBP
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- You have a daily supply of:
  - glass: 20 units

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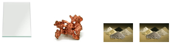

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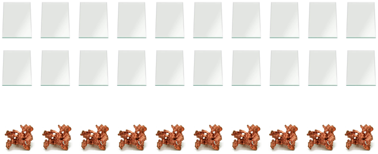

- You have a daily supply of:
  - glass: 20 units
  - copper: 10 units

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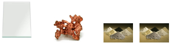

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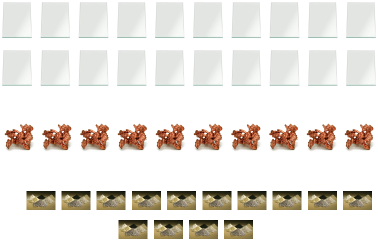

- You have a daily supply of:
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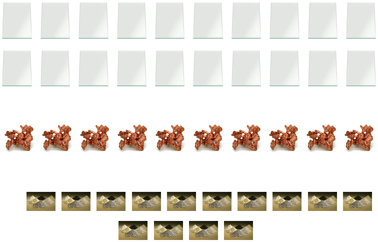

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    - (and enough of everything else...)

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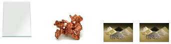

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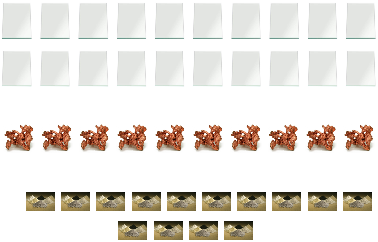

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How to maximise your daily earnings?

The Linear Program

Linear Program for the Production Problem

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The solution of this linear program yields the optimal production schedule.

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Formal Definition of Linear Program

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$$f(x_1, x_2, \ldots, x_n) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.$$

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Linear Constraints

- Linear-Progamming Problem: either minimise or maximise a linear function subject to a set of linear constraints

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Any setting of x_1 and x_2 satisfying all constraints is a feasible solution

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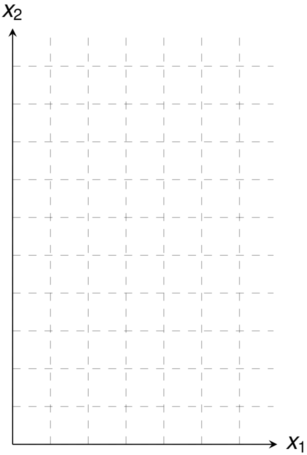

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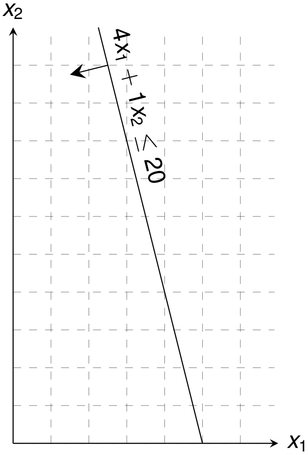

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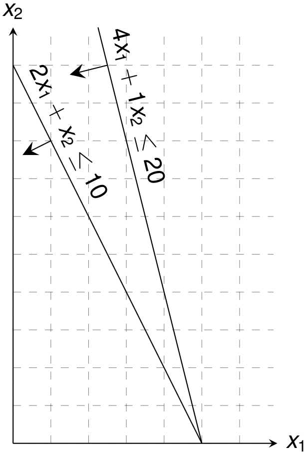

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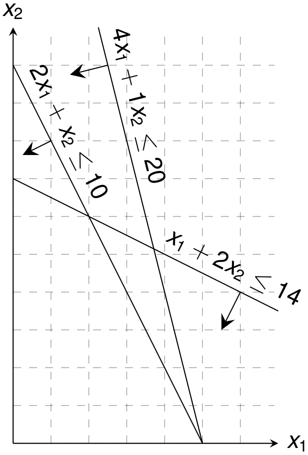

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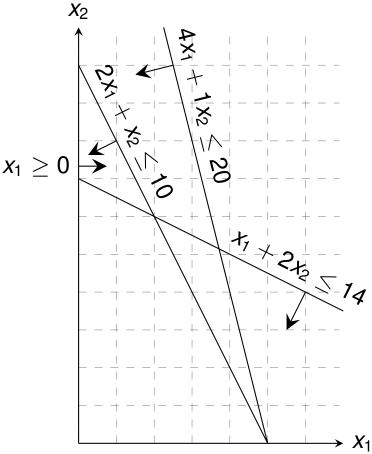

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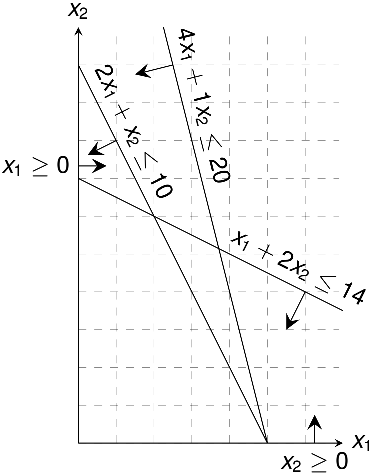

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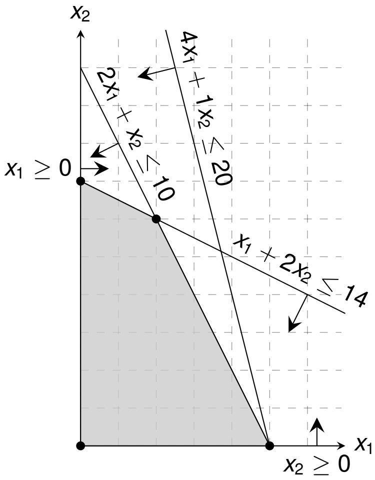

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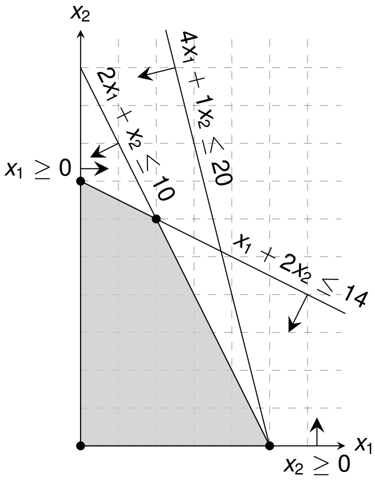
Graphical Procedure: Move the line $x_1 + x_2 = z$ as far up as possible.



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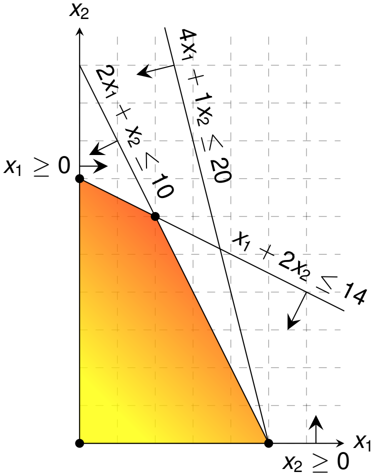

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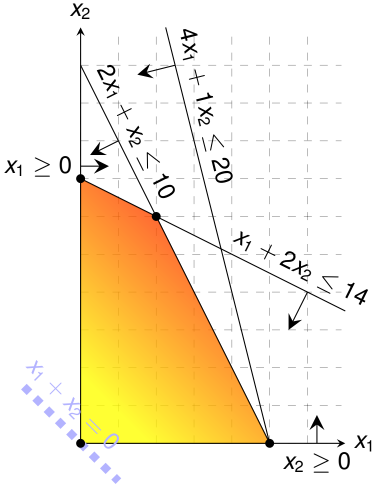

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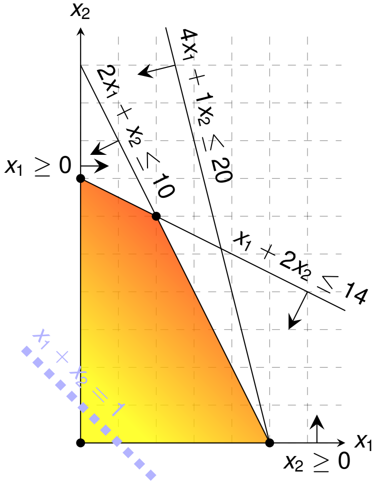

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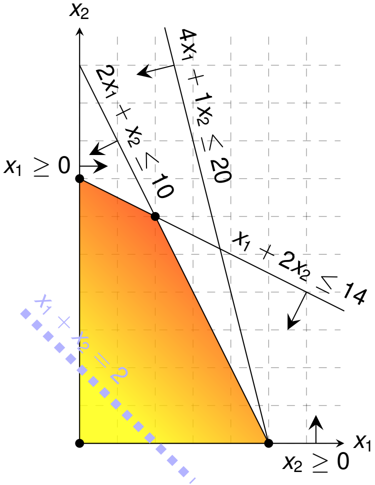

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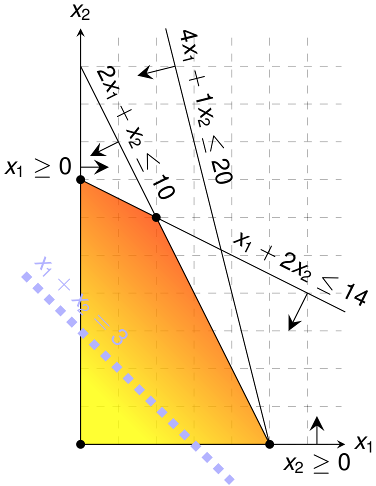

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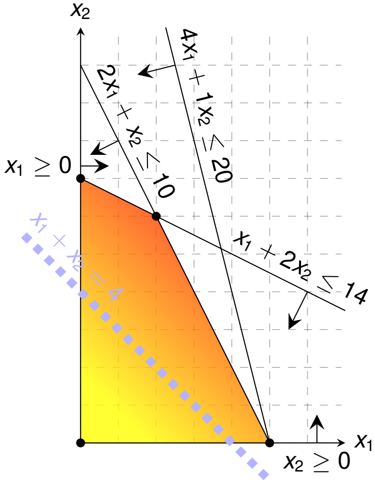
 & 2x_1 & + & x_2 & \leq & 10 \\
 & x_1 & + & 2x_2 & \leq & 14 \\
 & x_1, x_2 & & & \geq & 0
\end{array}
$$

Graphical Procedure: Move the line $x_1 + x_2 = z$ as far up as possible.



Finding the Optimal Production Schedule

$$
\begin{array}{lrcrcr}
\text{maximise} & x_1 & + & x_2 & & \\
\text{subject to} & & & & & \\
 & 4x_1 & + & x_2 & \leq & 20 \\
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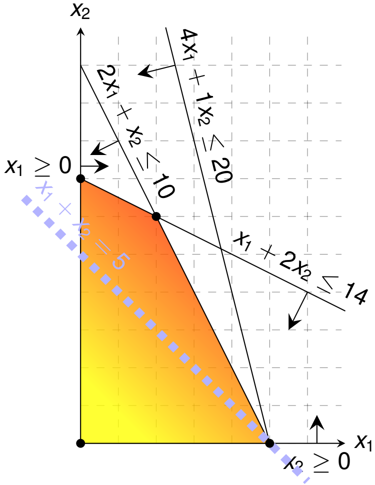

Finding the Optimal Production Schedule

$$
\begin{array}{llcrcl}
\text{maximise} & x_1 & + & x_2 & & \\
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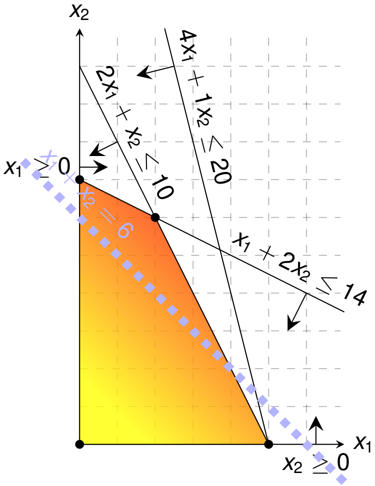

Finding the Optimal Production Schedule

$$\begin{array}{llcllcl} \text{maximise} & x_1 & + & x_2 & & \\ \text{subject to} & & & & & \\ & 4x_1 & + & x_2 & \leq & 20 \\ & 2x_1 & + & x_2 & \leq & 10 \\ & x_1 & + & 2x_2 & \leq & 14 \\ & x_1, x_2 & & & \geq & 0 \end{array}$$

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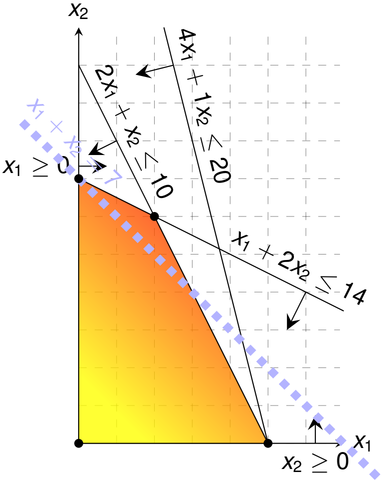

Finding the Optimal Production Schedule

$$\begin{array}{llcrcr}
\text{maximise} & x_1 & + & x_2 & & \\
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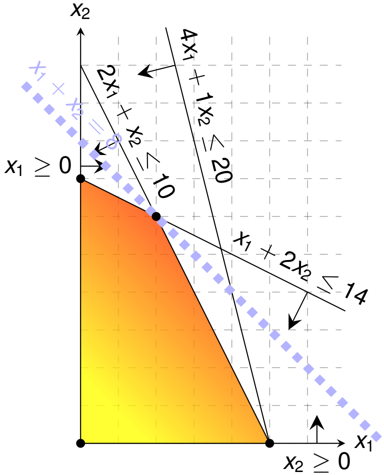

Graphical Procedure: Move the line $x_1 + x_2 = z$ as far up as possible.

Finding the Optimal Production Schedule

$$\begin{array}{llcccl} \text{maximise} & x_1 & + & x_2 & & \\ \text{subject to} & & & & & \\ & 4x_1 & + & x_2 & \leq & 20 \\ & 2x_1 & + & x_2 & \leq & 10 \\ & x_1 & + & 2x_2 & \leq & 14 \\ & x_1, x_2 & & & \geq & 0 \end{array}$$

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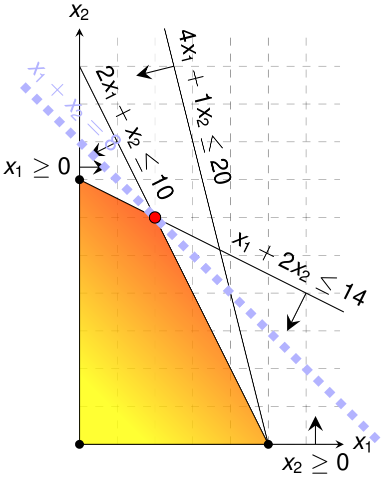

Finding the Optimal Production Schedule

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\begin{array}{lrcrcr}
\text{maximise} & x_1 & + & x_2 & & \\
\text{subject to} & & & & & \\
 & 4x_1 & + & x_2 & \leq & 20 \\
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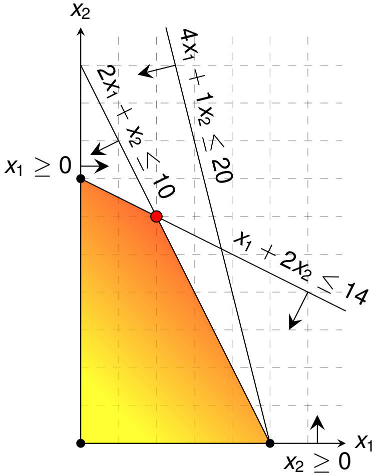

Graphical Procedure: Move the line $x_1 + x_2 = z$ as far up as possible.

Question: Which aspect did we ignore in the formulation of the linear program?

Finding the Optimal Production Schedule

$$
\begin{array}{llcrcr}
\text{maximise} & x_1 & + & x_2 & & \\
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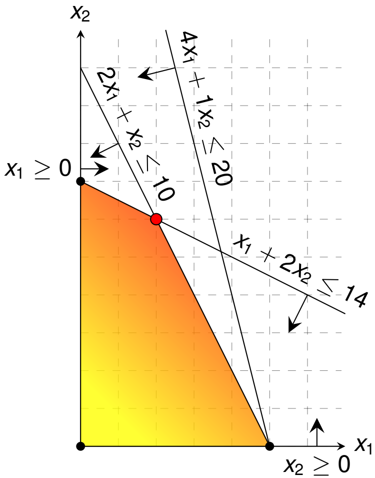


While the same approach also works for higher-dimensions, we need to take a more systematic and algebraic procedure.

Outline

Introduction

A Simple Example of a Linear Program

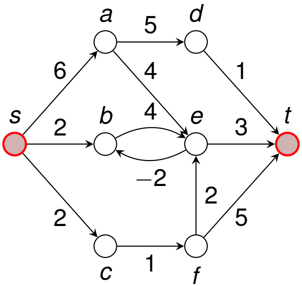
Formulating Problems as Linear Programs

Standard and Slack Forms

Shortest Paths

Single-Pair Shortest Path Problem

- Given: directed graph $G = (V, E)$ with edge weights $w : E \to \mathbb{R}$, pair of vertices $s, t \in V$



Shortest Paths

Single-Pair Shortest Path Problem

- Given: directed graph $G = (V, E)$ with edge weights $w : E \to \mathbb{R}$, pair of vertices $s, t \in V$
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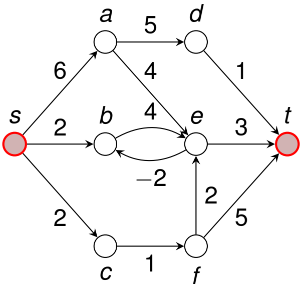

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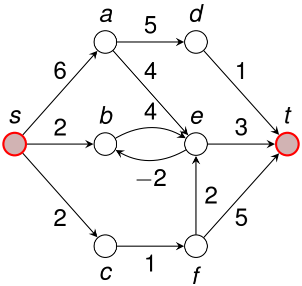

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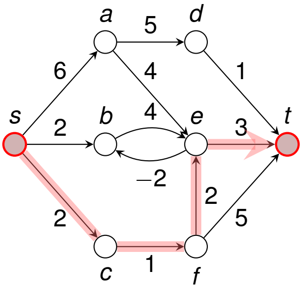

Shortest Paths

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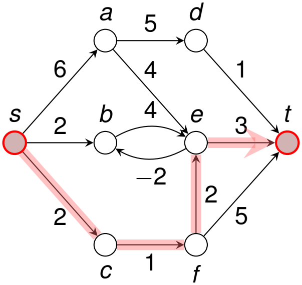
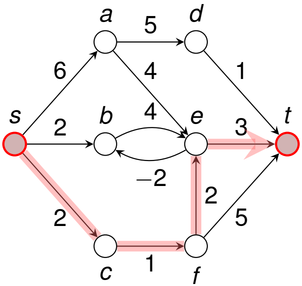


Exercise: How can we translate the SPSP problem into a linear program?

Shortest Paths



Single-Pair Shortest Path Problem

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Shortest Paths as LP

subject to

Shortest Paths

Single-Pair Shortest Path Problem

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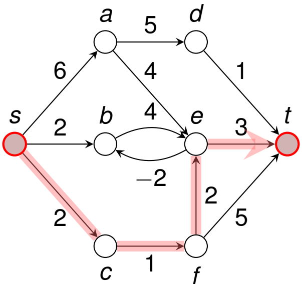
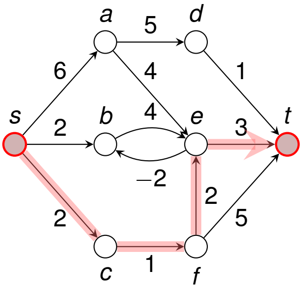


Shortest Paths as LP

subject to

$$\begin{array}{llllll} d_v & \leq & d_u & + & w(u,v) & \text{for each edge } (u,v) \in E, \\ d_s & = & 0. & & & \end{array}$$

Shortest Paths



Single-Pair Shortest Path Problem

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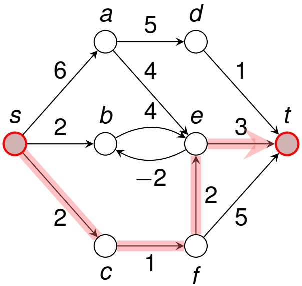
Shortest Paths as LP

$$\begin{array}{llllll} \text{maximise} & d_t & & & & \\ \text{subject to} & & & & & \\ & d_v & \leq & d_u & + \; w(u,v) & \text{for each edge } (u,v) \in E, \\ & d_s & = & 0. & & \end{array}$$

Shortest Paths



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this is a maximisation problem!

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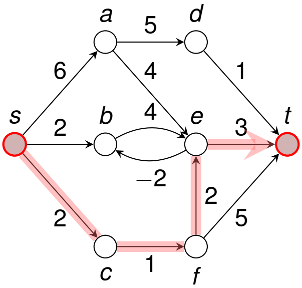

Shortest Paths as LP

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Recall: When BELLMAN-FORD terminates, all these inequalities are satisfied.

this is a maximisation problem!

Shortest Paths

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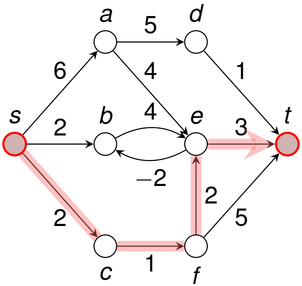

Shortest Paths as LP

$$
\begin{array}{lllll}
\text{maximise} & d_t & & & \\
\text{subject to} & & & & \\
& d_v & \leq & d_u + w(u,v) & \text{for each edge } (u,v) \in E, \\
& d_s & = & 0. &
\end{array}
$$

Recall: When BELLMAN-FORD terminates, all these inequalities are satisfied.

this is a maximisation problem!

Solution $\overline{d}$ satisfies $\overline{d}_v = \min_{u\colon (u,v)\in E} \left\{ \overline{d}_u + w(u,v) \right\}$

Maximum Flow

Maximum Flow Problem

- Given: directed graph $G = (V, E)$ with edge capacities $c : E \to \mathbb{R}^+$ (recall $c(u, v) = 0$ if $(u, v) \notin E$), pair of vertices $s, t \in V$

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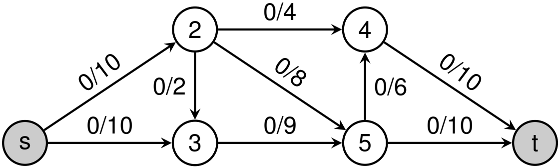

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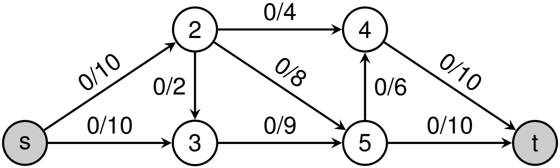
- Goal: Find a maximum flow $f : V \times V \to \mathbb{R}$ from s to t which satisfies the capacity constraints and flow conservation



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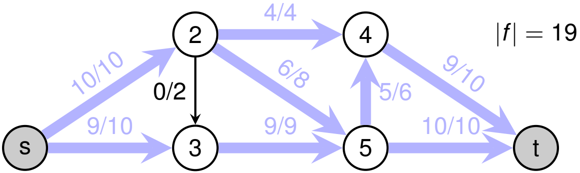

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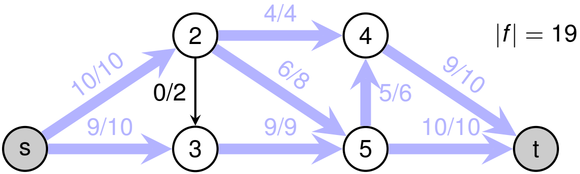

Maximum Flow as LP

$$
\begin{array}{llcll}
\text{maximise} & \sum_{v \in V} f_{sv} & - & \sum_{v \in V} f_{vs} & \\
\text{subject to} & & & & \\
& f_{uv} & \leq & c(u,v) & \text{for each } u, v \in V, \\
& \sum_{v \in V} f_{vu} & = & \sum_{v \in V} f_{uv} & \text{for each } u \in V \setminus \{s, t\}, \\
& f_{uv} & \geq & 0 & \text{for each } u, v \in V.
\end{array}
$$

Minimum-Cost Flow

Extension of the Maximum Flow Problem

Minimum-Cost-Flow Problem

Minimum-Cost Flow

Extension of the Maximum Flow Problem

Minimum-Cost-Flow Problem

- Given: directed graph $G = (V, E)$ with capacities $c : E \to \mathbb{R}^+$, pair of vertices $s, t \in V$, cost function $a : E \to \mathbb{R}^+$, flow demand of d units

Minimum-Cost Flow

Extension of the Maximum Flow Problem

Minimum-Cost-Flow Problem

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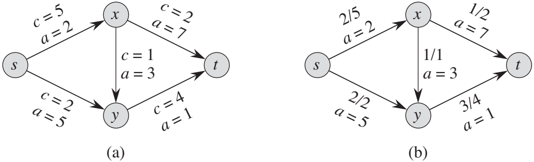

Figure 29.3 **(a)** An example of a minimum-cost-flow problem. We denote the capacities by c and the costs by a. Vertex s is the source and vertex t is the sink, and we wish to send 4 units of flow from s to t. **(b)** A solution to the minimum-cost flow problem in which 4 units of flow are sent from s to t. For each edge, the flow and capacity are written as flow/capacity.

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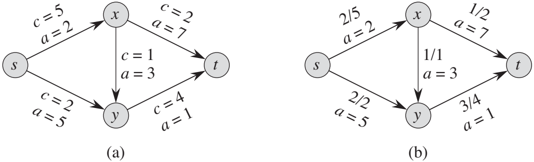
Optimal Solution with total cost:
$\sum_{(u,v)\in E} a(u,v) f_{uv} = (2\cdot 2)+(5\cdot 2)+(3\cdot 1)+(7\cdot 1)+(1\cdot 3) = 27$



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Minimum Cost Flow as a LP

Minimum Cost Flow as LP

minimise $\sum_{(u,v)\in E} a(u,v) f_{uv}$

subject to

$$\begin{aligned}
f_{uv} &\leq c(u,v) && \text{for } u,v \in V, \\
\sum_{v\in V} f_{vu} - \sum_{v\in V} f_{uv} &= 0 && \text{for } u \in V\setminus\{s,t\}, \\
\sum_{v\in V} f_{sv} - \sum_{v\in V} f_{vs} &= d\,, \\
f_{uv} &\geq 0 && \text{for } u,v \in V.
\end{aligned}$$

Minimum Cost Flow as a LP

Minimum Cost Flow as LP

minimise $\sum_{(u,v)\in E} a(u,v) f_{uv}$

subject to

$$\begin{aligned} f_{uv} &\leq c(u,v) && \text{for } u,v \in V, \\ \sum_{v\in V} f_{vu} - \sum_{v\in V} f_{uv} &= 0 && \text{for } u \in V \setminus \{s,t\}, \\ \sum_{v\in V} f_{sv} - \sum_{v\in V} f_{vs} &= d\,, \\ f_{uv} &\geq 0 && \text{for } u,v \in V. \end{aligned}$$

Real power of Linear Programming comes from the ability to solve **new problems**!

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Standard and Slack Forms

Standard Form

maximise $$\sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad \text{for } i = 1, 2, \ldots, m$$

$$x_j \geq 0 \qquad \text{for } j = 1, 2, \ldots, n$$

Standard and Slack Forms

Standard Form

maximise $$\sum_{j=1}^{n} c_j x_j$$

Objective Function

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad \text{for } i = 1, 2, \ldots, m$$

$$x_j \geq 0 \qquad \text{for } j = 1, 2, \ldots, n$$

Standard and Slack Forms

Standard Form

maximise $\sum_{j=1}^{n} c_j x_j$

Objective Function

subject to

$n + m$ constraints

$$\left\{ \begin{aligned} \sum_{j=1}^{n} a_{ij} x_j &\leq b_i && \text{for } i = 1, 2, \ldots, m \\ x_j &\geq 0 && \text{for } j = 1, 2, \ldots, n \end{aligned} \right.$$

Standard and Slack Forms

Standard Form

$$\text{maximise} \quad \sum_{j=1}^{n} c_j x_j$$

Objective Function

subject to

$$\left\{ \begin{aligned} \sum_{j=1}^{n} a_{ij} x_j \leq b_i & \qquad \text{for } i = 1, 2, \ldots, m \\ x_j \geq 0 & \qquad \text{for } j = 1, 2, \ldots, n \end{aligned} \right.$$

$n + m$ constraints

Non-Negativity Constraints

Standard and Slack Forms

Standard Form

maximise $\sum_{j=1}^{n} c_j x_j$

Objective Function

subject to

$$\left\{\begin{array}{ll} \sum_{j=1}^{n} a_{ij} x_j \leq b_i & \text{for } i = 1, 2, \ldots, m \\ x_j \geq 0 & \text{for } j = 1, 2, \ldots, n \end{array}\right.$$

$n + m$ constraints

Non-Negativity Constraints

Standard Form (Matrix-Vector-Notation)

maximise $c^T x$

Inner product of two vectors

subject to

$$Ax \leq b$$

Matrix-vector product

$$x \geq 0$$

Converting Linear Programs into Standard Form

Reasons for a LP not being in standard form:

1. The objective might be a minimisation rather than maximisation.
2. There might be variables without nonnegativity constraints.
3. There might be equality constraints.
4. There might be inequality constraints (with $\geq$ instead of $\leq$).

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Goal: Convert linear program into an equivalent program which is in standard form

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Goal: Convert linear program into an equivalent program which is in standard form

Equivalence: a correspondence (not necessarily a bijection) between solutions.

Converting into Standard Form (1/5)

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Converting into Standard Form (1/5)

Reasons for a LP not being in standard form:

1. The objective might be a minimisation rather than maximisation.

$$\begin{array}{lrcrcl}
\boxed{\text{minimise} \quad -2x_1 + 3x_2} & & & & & \\
\text{subject to} & & & & & \\
 & x_1 & + & x_2 & = & 7 \\
 & x_1 & - & 2x_2 & \leq & 4 \\
 & x_1 & & & \geq & 0
\end{array}$$

Converting into Standard Form (1/5)

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1. The objective might be a minimisation rather than maximisation.

$$\boxed{\text{minimise} \quad -2x_1 + 3x_2}$$

$$\begin{array}{lrcrcl}
\text{subject to} & & & & & \\
 & x_1 & + & x_2 & = & 7 \\
 & x_1 & - & 2x_2 & \leq & 4 \\
 & x_1 & & & \geq & 0
\end{array}$$

$\downarrow$ Negate objective function

Converting into Standard Form (1/5)

Reasons for a LP not being in standard form:

1. The objective might be a minimisation rather than maximisation.

$$\begin{array}{llcrcl} \boxed{\text{minimise} \quad -2x_1 + 3x_2} & & & & & \\ \text{subject to} & & & & & \\ & x_1 & + & x_2 & = & 7 \\ & x_1 & - & 2x_2 & \leq & 4 \\ & x_1 & & & \geq & 0 \end{array}$$

$\Big\downarrow$ Negate objective function

$$\begin{array}{llcrcl} \boxed{\text{maximise} \quad 2x_1 - 3x_2} & & & & & \\ \text{subject to} & & & & & \\ & x_1 & + & x_2 & = & 7 \\ & x_1 & - & 2x_2 & \leq & 4 \\ & x_1 & & & \geq & 0 \end{array}$$

Converting into Standard Form (2/5)

Reasons for a LP not being in standard form:

2. There might be variables without nonnegativity constraints.

Converting into Standard Form (2/5)

Reasons for a LP not being in standard form:

2. There might be variables without nonnegativity constraints.

$$
\begin{array}{lrcrcr}
\text{maximise} & 2x_1 & - & 3x_2 & & \\
\text{subject to} & & & & & \\
 & x_1 & + & x_2 & = & 7 \\
 & x_1 & - & 2x_2 & \leq & 4 \\
 & \boxed{x_1} & & & \boxed{\geq} & \boxed{0}
\end{array}
$$

Converting into Standard Form (2/5)

Reasons for a LP not being in standard form:

2. There might be variables without nonnegativity constraints.

$$\begin{array}{lcccccc} \text{maximise} & 2x_1 & - & 3x_2 & & \\ \text{subject to} & & & & & \\ & x_1 & + & x_2 & = & 7 \\ & x_1 & - & 2x_2 & \leq & 4 \\ & \multicolumn{5}{c}{\boxed{x_1 \qquad\qquad\qquad\qquad \geq \quad 0}} \end{array}$$

Replace x_2 by two non-negative variables x_2' and x_2''

Converting into Standard Form (2/5)

Reasons for a LP not being in standard form:

2. There might be variables without nonnegativity constraints.

$$
\begin{array}{ll}
\text{maximise} & 2x_1 - 3x_2 \\
\text{subject to} & \\
& x_1 + x_2 = 7 \\
& x_1 - 2x_2 \leq 4 \\
& \boxed{x_1 \geq 0}
\end{array}
$$

Replace x_2 by two non-negative variables x_2' and x_2''

$$
\begin{array}{ll}
\text{maximise} & 2x_1 - \boxed{3x_2' + 3x_2''} \\
\text{subject to} & \\
& x_1 + \boxed{x_2' - x_2''} = 7 \\
& x_1 - \boxed{2x_2' + 2x_2''} \leq 4 \\
& \boxed{x_1, x_2', x_2'' \geq 0}
\end{array}
$$

Converting into Standard Form (3/5)

Reasons for a LP not being in standard form:

3. There might be equality constraints.

Converting into Standard Form (3/5)

Reasons for a LP not being in standard form:

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$$\begin{array}{llllllll}
\text{maximise} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
\text{subject to} & & & & & & & \\
 & \boxed{x_1} & \boxed{+} & \boxed{x_2'} & \boxed{-} & \boxed{x_2''} & \boxed{=} & \boxed{7} \\
 & x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
 & \multicolumn{5}{l}{x_1, x_2', x_2''} & \geq & 0
\end{array}$$

Converting into Standard Form (3/5)

Reasons for a LP not being in standard form:

3. There might be equality constraints.

$$\begin{array}{ll} \text{maximise} & 2x_1 - 3x_2' + 3x_2'' \\ \text{subject to} & \\ & \boxed{x_1 + x_2' - x_2'' = 7} \\ & x_1 - 2x_2' + 2x_2'' \leq 4 \\ & x_1, x_2', x_2'' \geq 0 \end{array}$$

$\downarrow$ Replace each equality by two inequalities.

Converting into Standard Form (3/5)

Reasons for a LP not being in standard form:

3. There might be equality constraints.

$$
\begin{array}{ll}
\text{maximise} & 2x_1 - 3x_2' + 3x_2'' \\
\text{subject to} & \\
 & \boxed{\begin{array}{rcrcrcl} x_1 & + & x_2' & - & x_2'' & = & 7 \end{array}} \\
 & \begin{array}{rcrcrcl} x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \end{array} \\
 & x_1, x_2', x_2'' \geq 0
\end{array}
$$

Replace each equality by two inequalities.

$$
\begin{array}{ll}
\text{maximise} & 2x_1 - 3x_2' + 3x_2'' \\
\text{subject to} & \\
 & \boxed{\begin{array}{rcrcrcl} x_1 & + & x_2' & - & x_2'' & \leq & 7 \\ x_1 & + & x_2' & - & x_2'' & \geq & 7 \end{array}} \\
 & \begin{array}{rcrcrcl} x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \end{array} \\
 & x_1, x_2', x_2'' \geq 0
\end{array}
$$

Converting into Standard Form (4/5)

Reasons for a LP not being in standard form:

4. There might be inequality constraints (with $\geq$ instead of $\leq$).

Converting into Standard Form (4/5)

Reasons for a LP not being in standard form:

4. There might be inequality constraints (with $\geq$ instead of $\leq$).

$$\begin{array}{ll}
\text{maximise} & 2x_1 - 3x_2' + 3x_2'' \\
\text{subject to} & \\
& x_1 + x_2' - x_2'' \leq 7 \\
& \boxed{x_1 + x_2' - x_2'' \geq 7} \\
& x_1 - 2x_2' + 2x_2'' \leq 4 \\
& x_1, x_2', x_2'' \geq 0
\end{array}$$

Converting into Standard Form (4/5)

Reasons for a LP not being in standard form:

4. There might be inequality constraints (with $\geq$ instead of $\leq$).

$$
\begin{array}{lrcrcrcr}
\text{maximise} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
\text{subject to} & & & & & & & \\
 & x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
 & \multicolumn{7}{c}{\boxed{x_1 + x_2' - x_2'' \geq 7}} \\
 & x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
 & \multicolumn{5}{c}{x_1, x_2', x_2''} & \geq & 0
\end{array}
$$

$$\big\downarrow$$

Negate respective inequalities.

Converting into Standard Form (4/5)

Reasons for a LP not being in standard form:

4. There might be inequality constraints (with $\geq$ instead of $\leq$).

$$\begin{array}{ll} \text{maximise} & 2x_1 - 3x_2' + 3x_2'' \\ \text{subject to} & \\ & x_1 + x_2' - x_2'' \leq 7 \\ & \boxed{x_1 + x_2' - x_2'' \geq 7} \\ & x_1 - 2x_2' + 2x_2'' \leq 4 \\ & x_1, x_2', x_2'' \geq 0 \end{array}$$

$\Big\downarrow$ Negate respective inequalities.

$$\begin{array}{ll} \text{maximise} & 2x_1 - 3x_2' + 3x_2'' \\ \text{subject to} & \\ & x_1 + x_2' - x_2'' \leq 7 \\ & \boxed{-x_1 - x_2' + x_2'' \leq -7} \\ & x_1 - 2x_2' + 2x_2'' \leq 4 \\ & x_1, x_2', x_2'' \geq 0 \end{array}$$

Converting into Standard Form (5/5)

$$\begin{array}{lrcrcrcr}
\text{maximise} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & - & x_3 & \leq & 7 \\
& -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
& x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
& & x_1, x_2, x_3 & & & & \geq & 0
\end{array}$$

Converting into Standard Form (5/5)

Rename variable names (for consistency).

$$\begin{array}{lrcrcrcr}
\text{maximise} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & - & x_3 & \leq & 7 \\
& -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
& x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
& \multicolumn{5}{c}{x_1, x_2, x_3} & \geq & 0
\end{array}$$

Converting into Standard Form (5/5)

Rename variable names (for consistency).

$$\begin{array}{lrcrcrcr}
\text{maximise} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & - & x_3 & \leq & 7 \\
& -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
& x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
& & & x_1, x_2, x_3 & & & \geq & 0
\end{array}$$

It is always possible to convert a linear program into standard form.

Converting Standard Form into Slack Form (1/3)

Goal: Convert standard form into slack form, where all constraints except for the non-negativity constraints are equalities.

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For the simplex algorithm, it is more convenient to work with equality constraints.

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Introducing Slack Variables

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Introducing Slack Variables

- Let $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ be an inequality constraint

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- Introduce a slack variable s by

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- Let $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ be an inequality constraint
- Introduce a slack variable s by

$$s = b_i - \sum_{j=1}^{n} a_{ij} x_j$$

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$$s \geq 0.$$

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$$s = b_i - \sum_{j=1}^{n} a_{ij}x_j$$
$$s \geq 0.$$

s measures the slack between the two sides of the inequality.

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- Let $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ be an inequality constraint
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$$s = b_i - \sum_{j=1}^{n} a_{ij}x_j$$

$$s \geq 0.$$

s measures the slack between the two sides of the inequality.

- Denote slack variable of the i-th inequality by x_{n+i}

Converting Standard Form into Slack Form (2/3)

$$\begin{array}{lrcrcrcr}
\text{maximise} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
 & x_1 & + & x_2 & - & x_3 & \leq & 7 \\
 & -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
 & x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
 & \multicolumn{5}{c}{x_1, x_2, x_3} & \geq & 0
\end{array}$$

Converting Standard Form into Slack Form (2/3)

$$
\begin{array}{lrcrcrcr}
\text{maximise} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
 & x_1 & + & x_2 & - & x_3 & \leq & 7 \\
 & -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
 & x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
 & \multicolumn{5}{c}{x_1, x_2, x_3} & \geq & 0
\end{array}
$$

Introduce slack variables

Converting Standard Form into Slack Form (2/3)

$$
\begin{array}{lrcrcrcr}
\text{maximise} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
 & x_1 & + & x_2 & - & x_3 & \leq & 7 \\
 & -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
 & x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
 & & x_1, x_2, x_3 & & & & \geq & 0
\end{array}
$$

$$\Big\downarrow \; \text{Introduce slack variables}$$

$$
\begin{array}{lrcrcrcrcr}
\text{subject to} & & & & & & & & & \\
 & x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3
\end{array}
$$

Converting Standard Form into Slack Form (2/3)

$$\begin{array}{lrcrcrcr}
\text{maximise} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
 & x_1 & + & x_2 & - & x_3 & \leq & 7 \\
 & -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
 & x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
 & \multicolumn{5}{c}{x_1, x_2, x_3} & \geq & 0
\end{array}$$

$$\Big\downarrow$$

Introduce slack variables

subject to

$$\begin{array}{rcrcrcrcr}
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3
\end{array}$$

Converting Standard Form into Slack Form (2/3)

$$
\begin{array}{lrcrcrcr}
\text{maximise} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
 & x_1 & + & x_2 & - & x_3 & \leq & 7 \\
 & -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
 & x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
 & \multicolumn{5}{c}{x_1, x_2, x_3} & \geq & 0
\end{array}
$$

Introduce slack variables

subject to

$$
\begin{array}{rcrcrcrcr}
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}
$$

Converting Standard Form into Slack Form (2/3)

$$
\begin{array}{lrcrcrcr}
\text{maximise} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
 & x_1 & + & x_2 & - & x_3 & \leq & 7 \\
 & -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
 & x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
 & \multicolumn{5}{c}{x_1, x_2, x_3} & \geq & 0
\end{array}
$$

Introduce slack variables

$$
\begin{array}{lrcrcrcrcr}
\text{subject to} & & & & & & & & & \\
 & x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
 & x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
 & x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
 & \multicolumn{5}{c}{x_1, x_2, x_3, x_4, x_5, x_6} & \geq & 0 & &
\end{array}
$$

Converting Standard Form into Slack Form (2/3)

$$\begin{array}{lrcrcrcr}
\text{maximise} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & - & x_3 & \leq & 7 \\
& -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
& x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
& & x_1, x_2, x_3 & & & & \geq & 0
\end{array}$$

$$\Big\downarrow \text{ Introduce slack variables}$$

$$\begin{array}{lrcrcrcr}
\text{maximise} & 2x_1 & - & 3x_2 & + & 3x_3 & & & & \\
\text{subject to} & & & & & & & & & \\
& x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
& x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
& x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
& \multicolumn{4}{r}{x_1, x_2, x_3, x_4, x_5, x_6} & \geq & 0 & & &
\end{array}$$

Converting Standard Form into Slack Form (3/3)

$$\begin{array}{llcrcrcrcr}
\text{maximise} & 2x_1 - 3x_2 + 3x_3 \\
\text{subject to} \\
& x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
& x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
& x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
& \multicolumn{5}{c}{x_1, x_2, x_3, x_4, x_5, x_6} & \geq & 0
\end{array}$$

Converting Standard Form into Slack Form (3/3)

$$\begin{array}{lrcrcrcrcr}
\text{maximise} & 2x_1 & - & 3x_2 & + & 3x_3 \\
\text{subject to} \\
& x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
& x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
& x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
& \multicolumn{5}{c}{x_1, x_2, x_3, x_4, x_5, x_6} & \geq & 0
\end{array}$$

Use variable z to denote objective function and omit the nonnegativity constraints.

Converting Standard Form into Slack Form (3/3)

$$\begin{array}{ll} \text{maximise} & 2x_1 - 3x_2 + 3x_3 \\ \text{subject to} & \\ & \begin{array}{rcrcrcrcr} x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\ x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\ x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \end{array} \\ & \quad x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{array}$$

Use variable z to denote objective function and omit the nonnegativity constraints.

$$\begin{array}{rcrcrcrcr} \boxed{z} & \boxed{=} & & & \boxed{2x_1} & \boxed{-} & \boxed{3x_2} & \boxed{+} & \boxed{3x_3} \\ x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\ x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\ x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \end{array}$$

Converting Standard Form into Slack Form (3/3)

$$
\begin{array}{llrrrrrrrr}
\text{maximise} & 2x_1 & - & 3x_2 & + & 3x_3 \\
\text{subject to} \\
& x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
& x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
& x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
& \multicolumn{5}{r}{x_1, x_2, x_3, x_4, x_5, x_6} & \geq & 0
\end{array}
$$

Use variable z to denote objective function and omit the nonnegativity constraints.

$$
\boxed{z = 2x_1 - 3x_2 + 3x_3}
$$

$$
\begin{array}{lcrcrcrcr}
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}
$$

This is called slack form.

Basic and Non-Basic Variables

$$\begin{array}{rcrcrcrcr}
z & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}$$

Basic and Non-Basic Variables

$$\begin{array}{rcrcrcrcr} z & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\ x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\ x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\ x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \end{array}$$

Basic Variables: $B = \{4, 5, 6\}$

Basic and Non-Basic Variables

$$\begin{array}{rcrcrcrcr}
z & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}$$

Basic Variables: $B = \{4, 5, 6\}$

Non-Basic Variables: $N = \{1, 2, 3\}$

Basic and Non-Basic Variables

$$
\begin{array}{rclrrrrrrr}
z & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}
$$

Basic Variables: $B = \{4, 5, 6\}$

Non-Basic Variables: $N = \{1, 2, 3\}$

Slack Form (Formal Definition)

Slack form is given by a tuple (N, B, A, b, c, v) so that

$$z = v + \sum_{j \in N} c_j x_j$$

$$x_i = b_i - \sum_{j \in N} a_{ij} x_j \qquad \text{for } i \in B,$$

and all variables are non-negative.

Basic and Non-Basic Variables

$$
\begin{array}{rcrcrcrcr}
z & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}
$$

Basic Variables: $B = \{4, 5, 6\}$

Non-Basic Variables: $N = \{1, 2, 3\}$

Slack Form (Formal Definition)

Slack form is given by a tuple (N, B, A, b, c, v) so that

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$$x_i = b_i - \sum_{j \in N} a_{ij} x_j \qquad \text{for } i \in B,$$

and all variables are non-negative.

Variables/Coefficients on the right hand side are indexed by B and N.

Slack Form (Example)

$$\begin{array}{rcrcrcrcr}
z & = & 28 & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} \\
x_1 & = & 8 & + & \frac{x_3}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} \\
x_2 & = & 4 & - & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} \\
x_4 & = & 18 & - & \frac{x_3}{2} & + & \frac{x_5}{2} & &
\end{array}$$

Slack Form (Example)

$$\begin{array}{rcrcrcrcr}
z & = & 28 & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} \\
x_1 & = & 8 & + & \frac{x_3}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} \\
x_2 & = & 4 & - & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} \\
x_4 & = & 18 & - & \frac{x_3}{2} & + & \frac{x_5}{2} & &
\end{array}$$

Slack Form Notation

Slack Form (Example)

$$\begin{array}{rcrcrcrcr} z & = & 28 & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} \\ x_1 & = & 8 & + & \frac{x_3}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} \\ x_2 & = & 4 & - & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} \\ x_4 & = & 18 & - & \frac{x_3}{2} & + & \frac{x_5}{2} & & \end{array}$$

Slack Form Notation

- $B = \{1, 2, 4\}$, $N = \{3, 5, 6\}$

Slack Form (Example)

$$\begin{array}{rcrcrcrcr} z & = & 28 & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} \\ x_1 & = & 8 & + & \frac{x_3}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} \\ x_2 & = & 4 & - & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} \\ x_4 & = & 18 & - & \frac{x_3}{2} & + & \frac{x_5}{2} & & \end{array}$$

Slack Form Notation

- $B = \{1, 2, 4\}$, $N = \{3, 5, 6\}$
- $$A = \begin{pmatrix} a_{13} & a_{15} & a_{16} \\ a_{23} & a_{25} & a_{26} \\ a_{43} & a_{45} & a_{46} \end{pmatrix} = \begin{pmatrix} -1/6 & -1/6 & 1/3 \\ 8/3 & 2/3 & -1/3 \\ 1/2 & -1/2 & 0 \end{pmatrix}$$

Slack Form (Example)

$$\begin{array}{rclcrcrcr} z & = & 28 & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} \\ x_1 & = & 8 & + & \frac{x_3}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} \\ x_2 & = & 4 & - & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} \\ x_4 & = & 18 & - & \frac{x_3}{2} & + & \frac{x_5}{2} & & \end{array}$$

Slack Form Notation

- $B = \{1, 2, 4\}$, $N = \{3, 5, 6\}$
- $$A = \begin{pmatrix} a_{13} & a_{15} & a_{16} \\ a_{23} & a_{25} & a_{26} \\ a_{43} & a_{45} & a_{46} \end{pmatrix} = \begin{pmatrix} -1/6 & -1/6 & 1/3 \\ 8/3 & 2/3 & -1/3 \\ 1/2 & -1/2 & 0 \end{pmatrix}$$
- $$b = \begin{pmatrix} b_1 \\ b_2 \\ b_4 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 18 \end{pmatrix},$$

Slack Form (Example)

$$\begin{array}{rcrcccccc} z & = & 28 & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} \\ x_1 & = & 8 & + & \frac{x_3}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} \\ x_2 & = & 4 & - & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} \\ x_4 & = & 18 & - & \frac{x_3}{2} & + & \frac{x_5}{2} & & \end{array}$$

Slack Form Notation

- $B = \{1, 2, 4\}$, $N = \{3, 5, 6\}$
- $$A = \begin{pmatrix} a_{13} & a_{15} & a_{16} \\ a_{23} & a_{25} & a_{26} \\ a_{43} & a_{45} & a_{46} \end{pmatrix} = \begin{pmatrix} -1/6 & -1/6 & 1/3 \\ 8/3 & 2/3 & -1/3 \\ 1/2 & -1/2 & 0 \end{pmatrix}$$
- $$b = \begin{pmatrix} b_1 \\ b_2 \\ b_4 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 18 \end{pmatrix}, \quad c = \begin{pmatrix} c_3 \\ c_5 \\ c_6 \end{pmatrix} = \begin{pmatrix} -1/6 \\ -1/6 \\ -2/3 \end{pmatrix}$$

Slack Form (Example)

$$\begin{array}{rcrcccccc}
z & = & 28 & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} \\
x_1 & = & 8 & + & \frac{x_3}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} \\
x_2 & = & 4 & - & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} \\
x_4 & = & 18 & - & \frac{x_3}{2} & + & \frac{x_5}{2} & &
\end{array}$$

Slack Form Notation

- $B = \{1, 2, 4\}$, $N = \{3, 5, 6\}$
- $$A = \begin{pmatrix} a_{13} & a_{15} & a_{16} \\ a_{23} & a_{25} & a_{26} \\ a_{43} & a_{45} & a_{46} \end{pmatrix} = \begin{pmatrix} -1/6 & -1/6 & 1/3 \\ 8/3 & 2/3 & -1/3 \\ 1/2 & -1/2 & 0 \end{pmatrix}$$
- $$b = \begin{pmatrix} b_1 \\ b_2 \\ b_4 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 18 \end{pmatrix}, \quad c = \begin{pmatrix} c_3 \\ c_5 \\ c_6 \end{pmatrix} = \begin{pmatrix} -1/6 \\ -1/6 \\ -2/3 \end{pmatrix}$$
- $v = 28$