Randomised Algorithms

Lecture 10: Approximation Algorithms: Set-Cover and MAX-CNF

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Lent 2023

UNIVERSITY OF CAMBRIDGE

Outline

Weighted Set Cover

MAX-CNF

Appendix: An Approximation Algorithm of TSP (non-examin.)

The Weighted Set-Cover Problem

Set Cover Problem

- Given: set X and a family of subsets $\mathcal{F}$, and a cost function $c : \mathcal{F} \to \mathbb{R}^+$
- Goal: Find a minimum-cost subset $\mathcal{C} \subseteq \mathcal{F}$

$$\text{s.t.} \qquad X = \bigcup_{S \in \mathcal{C}} S.$$

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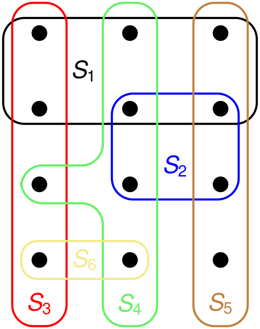
Sum over the costs of all sets in $\mathcal{C}$



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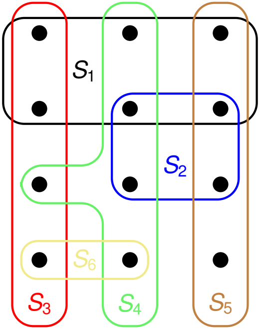

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c :	2	3	3	5	1	2

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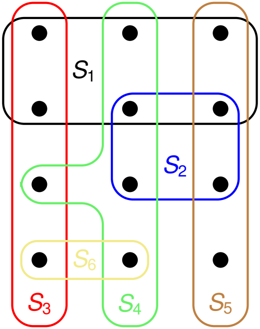

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Remarks:

- generalisation of the weighted Vertex-Cover problem
- models resource allocation problems

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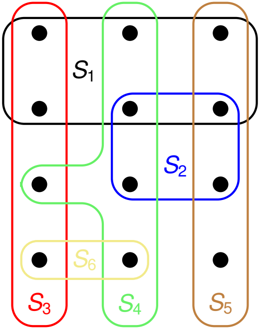

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Question: How can we reduce the Vertex-Cover problem to the Set-Cover problem?

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- generalisation of the weighted Vertex-Cover problem
- models resource allocation problems

Setting up an Integer Program

Exercise: Try to formulate the integer program and linear program of the weighted SET-COVER problem (solution on next slide!)

Setting up an Integer Program

0-1 Integer Program

$$
\begin{array}{llcll}
\text{minimize} & \displaystyle\sum_{S \in \mathcal{F}} c(S) y(S) & & & \\
\text{subject to} & \displaystyle\sum_{S \in \mathcal{F} \colon x \in S} y(S) & \geq & 1 & \text{for each } x \in X \\
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Linear Program

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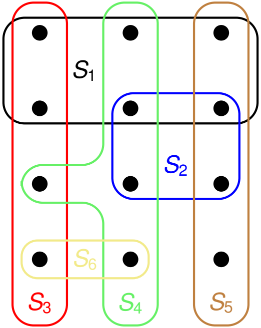
Back to the Example



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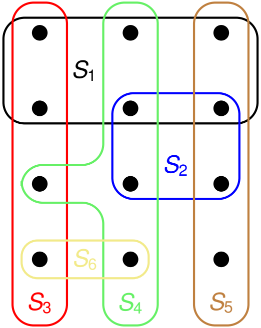

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$\overline{y}(.)$:	1/2	1/2	1/2	1/2	1	1/2

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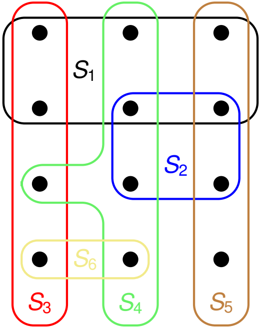

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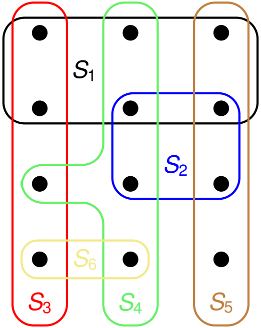

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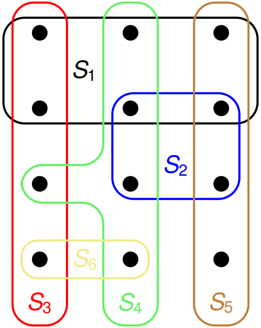

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The strategy employed for Vertex-Cover would take all 6 sets!

Even worse: If all $\overline{y}$'s were below $1/2$, we would not even return a valid cover!

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Idea: Interpret the $\overline{y}$-values as probabilities for picking the respective set.

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- Let $\mathcal{C} \subseteq \mathcal{F}$ be a random set with each set S being included independently with probability $\overline{y}(S)$.
- More precisely, if $\overline{y}$ denotes the optimal solution of the LP, then we compute an integral solution y by:

$$y(S) = \begin{cases} 1 & \text{with probability } \overline{y}(S) \\ 0 & \text{otherwise.} \end{cases} \qquad \text{for all } S \in \mathcal{F}.$$

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- Therefore, $\mathbf{E}\left[\, y(S) \,\right] = \overline{y}(S)$.

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- The probability that an element $x \in X$ is covered satisfies

$$\mathbf{P}\left[x \in \bigcup_{S \in \mathcal{C}} S \right] \geq 1 - \frac{1}{e}.$$

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Lemma

Let $\mathcal{C} \subseteq \mathcal{F}$ be a random subset with each set S being included independently with probability $\overline{y}(S)$.

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$$\leq \prod_{S \in \mathcal{F}:\ x \in S} e^{-\overline{y}(S)}$$

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$$\mathbf{E}[c(\mathcal{C})] = \mathbf{E}\left[\sum_{S \in \mathcal{C}} c(S)\right] = \mathbf{E}\left[\sum_{S \in \mathcal{F}} \mathbf{1}_{S \in \mathcal{C}} \cdot c(S)\right]$$
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- **Step 2**: The probability for an element to be (not) covered

$$\mathbf{P}[x \notin \cup_{S \in \mathcal{C}} S] = \prod_{S \in \mathcal{F}:\, x \in S} \mathbf{P}[S \notin \mathcal{C}] = \prod_{S \in \mathcal{F}:\, x \in S} (1 - \overline{y}(S))$$
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$1 + x \leq e^x$ for any $x \in \mathbb{R}$

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compute ȳ, an optimal solution to the linear program
C = ∅
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        let C = C ∪ {S} with probability ȳ(S)
return C
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WEIGHTED SET COVER-LP$(X, \mathcal{F}, c)$

1: compute $\overline{y}$, an optimal solution to the linear program
2: $\mathcal{C} = \emptyset$
3: **repeat** $2 \ln n$ times
4:     **for** each $S \in \mathcal{F}$
5:         let $\mathcal{C} = \mathcal{C} \cup \{S\}$ with probability $\overline{y}(S)$
6: **return** $\mathcal{C}$

clearly runs in polynomial-time!

Analysis of WEIGHTED SET COVER-LP

Theorem

- With probability at least $1 - \frac{1}{n}$, the returned set $\mathcal{C}$ is a valid cover of X.
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  - Linearity $\Rightarrow \mathbf{E}\left[\, c(\mathcal{C}) \,\right] \leq 2\ln(n)\cdot\sum_{S\in\mathcal{F}} c(S)\cdot\overline{y}(S)$

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- **Step 1**: The probability that $\mathcal{C}$ is a cover ✓
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[Exercise Question (9/10).10] gives a different perspective on the amplification procedure through non-linear randomised rounding.

Outline

Weighted Set Cover

MAX-CNF

Appendix: An Approximation Algorithm of TSP (non-examin.)

MAX-CNF

Recall:

MAX-3-CNF Satisfiability

- Given: 3-CNF formula, e.g.: $(x_1 \vee x_3 \vee \overline{x_4}) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_5}) \wedge \cdots$
- Goal: Find an assignment of the variables that satisfies as many clauses as possible.

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Why study this generalised problem?

- Allowing arbitrary clause lengths makes the problem more interesting (we will see that simply guessing is not the best!)
- a nice concluding example where we can practice previously learned approaches

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Assign each variable true or false uniformly and independently at random.

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For any clause i which has length ℓ,

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- First statement as in the proof of Theorem 35.6. For clause *i* not to be satisfied, all ℓ occurring variables must be set to a specific value.

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- First statement as in the proof of Theorem 35.6. For clause i not to be satisfied, all ℓ occurring variables must be set to a specific value.
- As before, let $Y := \sum_{i=1}^{m} Y_i$ be the number of satisfied clauses. Then,

$$\mathbf{E}[Y] = \mathbf{E}\left[\sum_{i=1}^{m} Y_i\right] = \sum_{i=1}^{m} \mathbf{E}[Y_i] \geq \sum_{i=1}^{m} \frac{1}{2} = \frac{1}{2} \cdot m. \qquad \square$$

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First solve a linear program and use fractional values for a **biased** coin flip.

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0-1 Integer Program

$$\begin{array}{llll} \text{maximize} & \sum_{i=1}^{m} z_i & & \\ \text{subject to} & \sum_{j \in C_i^+} y_j + \sum_{j \in C_i^-} (1 - y_j) \geq z_i & & \text{for each } i = 1, 2, \ldots, m \\ & z_i \in \{0, 1\} & & \text{for each } i = 1, 2, \ldots, m \\ & y_j \in \{0, 1\} & & \text{for each } j = 1, 2, \ldots, n \end{array}$$

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C_i^+ is the index set of the un-negated variables of clause i.

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These auxiliary variables are used to reflect whether a clause is satisfied or not

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- In the corresponding LP each $\in \{0, 1\}$ is replaced by $\in [0, 1]$
- Let $(\overline{y}, \overline{z})$ be the optimal solution of the LP
- Obtain an integer solution y through randomised rounding of $\overline{y}$

Analysis of Randomised Rounding

Lemma

For any clause i of length ℓ,

$$\mathbf{P}\left[\,\text{clause } i \text{ is satisfied}\,\right] \geq \left(1 - \left(1 - \frac{1}{\ell}\right)^{\ell}\right) \cdot \overline{z}_i.$$

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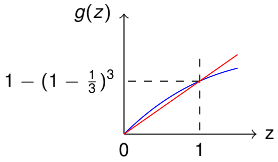

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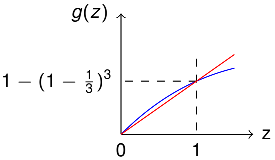

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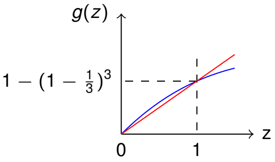

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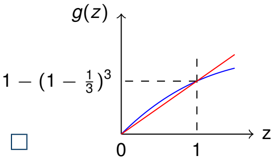

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LP solution at least as good as optimum

Approach 3: Hybrid Algorithm

Summary

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If $b = 0$ **then** perform random guessing
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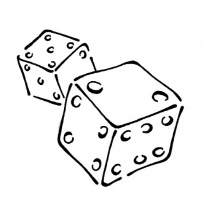
return the computed solution



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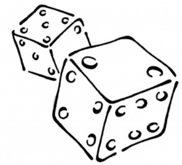

Algorithm sets each variable x_i to TRUE with prob. $\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \overline{y}_i$.
Note, however, that variables are **not** independently assigned!

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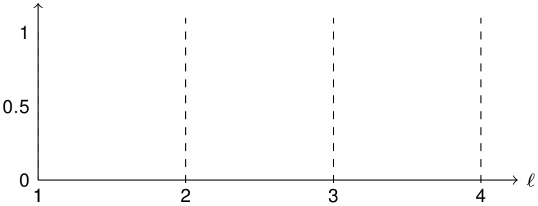

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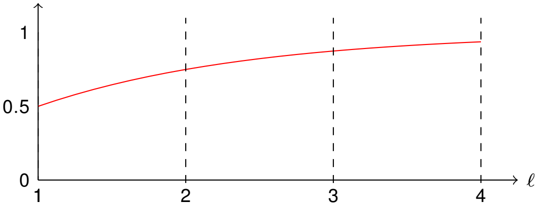

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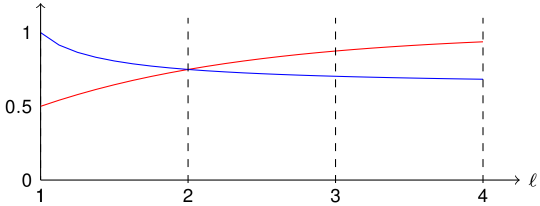

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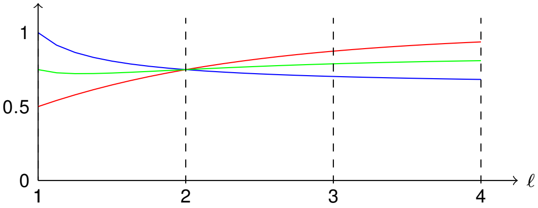

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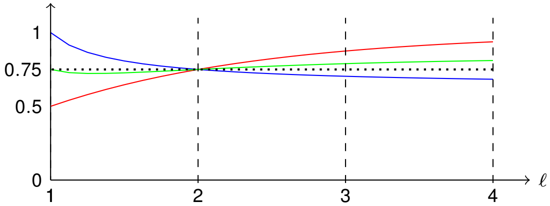

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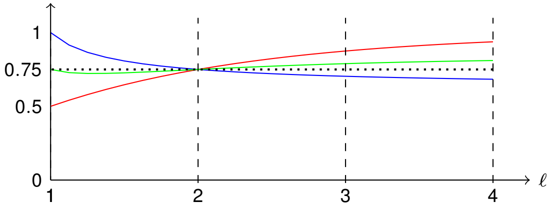

MAX-CNF Conclusion

Summary

- Since $\alpha_2 = \beta_2 = 3/4$, we cannot achieve a better approximation ratio than $4/3$ by combining Algorithm 1 & 2 in a different way
- The $4/3$-approximation algorithm can be easily derandomised
  - Idea: use the conditional expectation trick for both Algorithm 1 & 2 and output the better solution
- The $4/3$-approximation algorithm applies unchanged to a weighted version of MAX-CNF, where each clause has a non-negative weight
- Even MAX-2-CNF (every clause has length 2) is NP-hard!

Outline

Weighted Set Cover

MAX-CNF

Appendix: An Approximation Algorithm of TSP (non-examin.)

Metric TSP (TSP Problem with the Triangle Inequality)

Idea: First compute an MST, and then create a tour based on the tree.

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APPROX-TSP-TOUR(G, c)
select a vertex $r \in G.V$ to be a "root" vertex
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let H be a list of vertices, ordered according to when they are first visited
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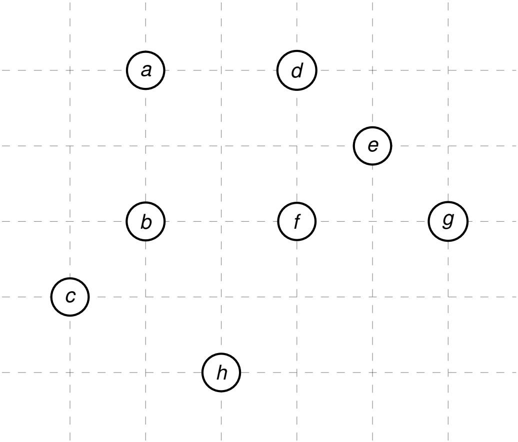
Remember: In the Metric-TSP problem, G is a complete graph.

Run of APPROX-TSP-TOUR



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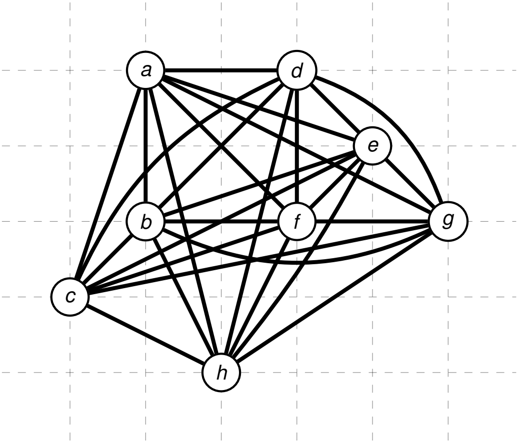

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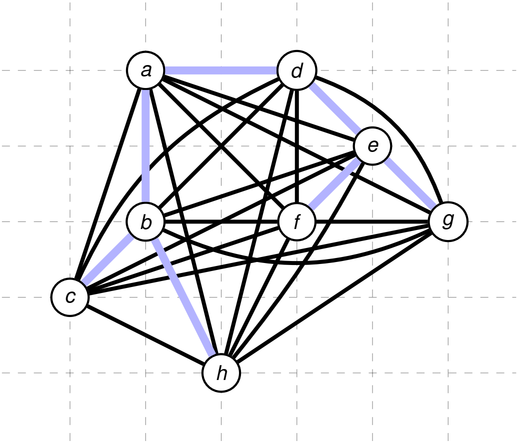
Run of APPROX-TSP-TOUR



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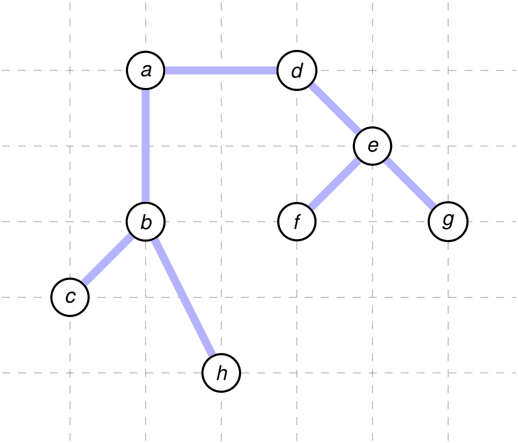

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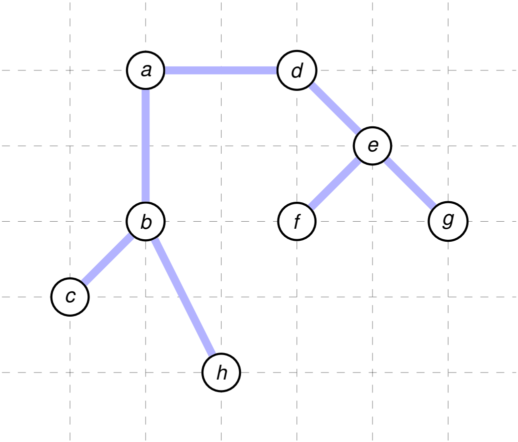

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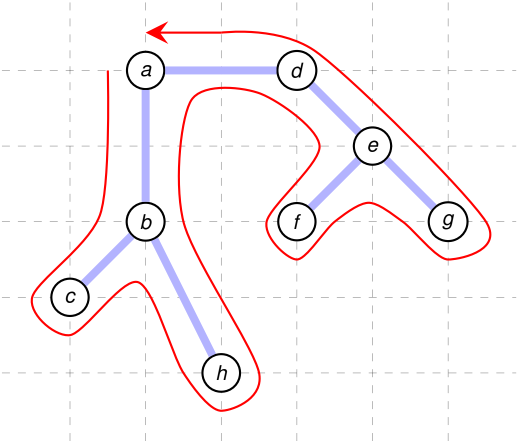

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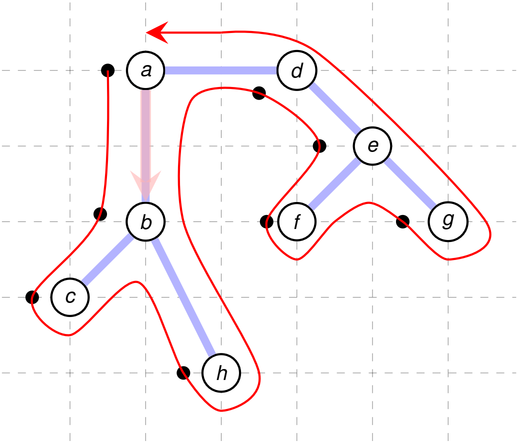

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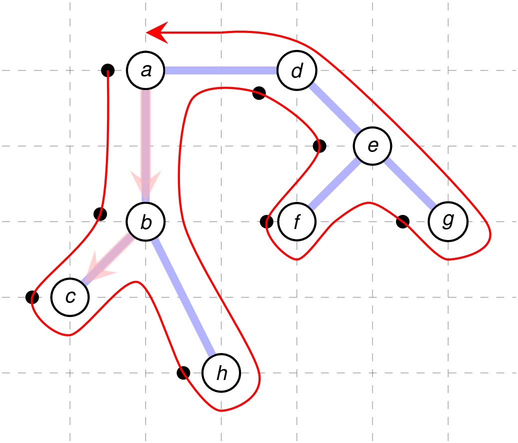
3. Return list of vertices according to the preorder tree walk

Run of APPROX-TSP-TOUR



1. Compute MST $T_{\min}$ ✓
2. Perform preorder walk on MST $T_{\min}$ ✓
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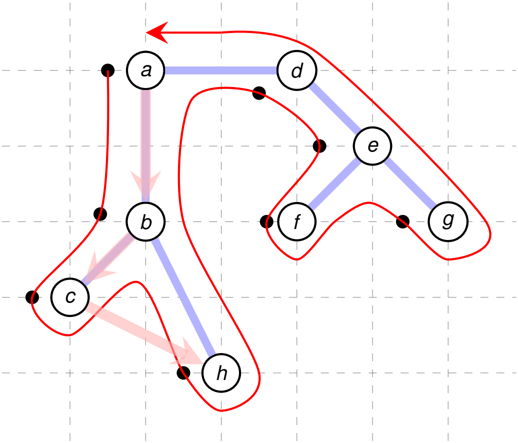

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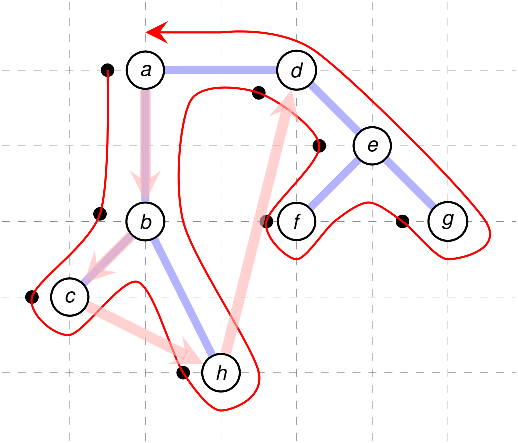

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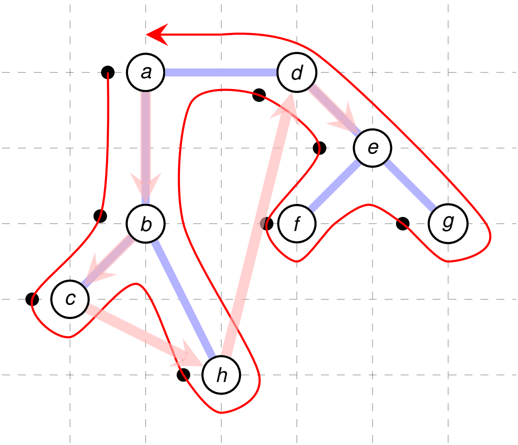
Run of Approx-Tsp-Tour



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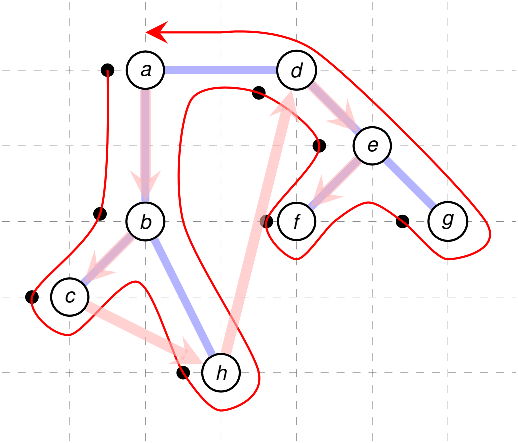
Run of APPROX-TSP-TOUR



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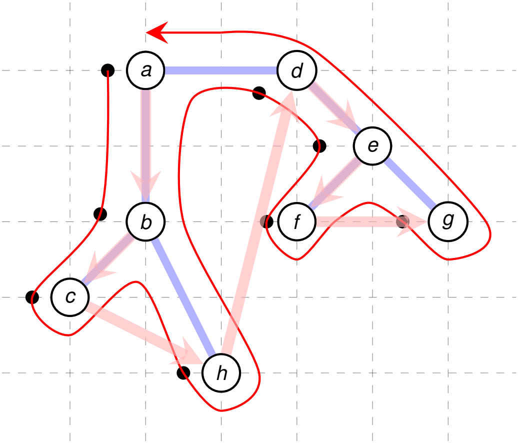

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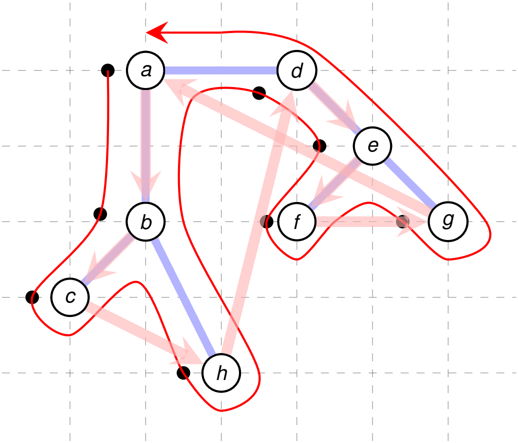

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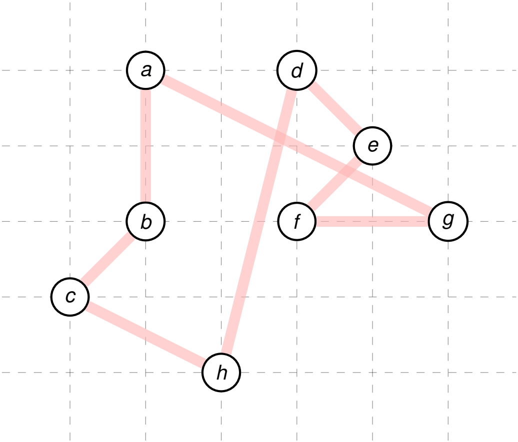

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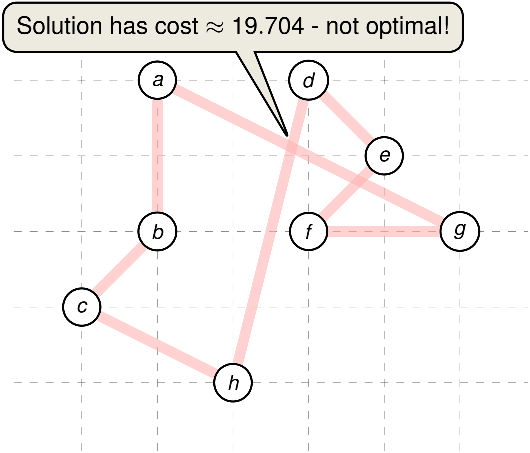

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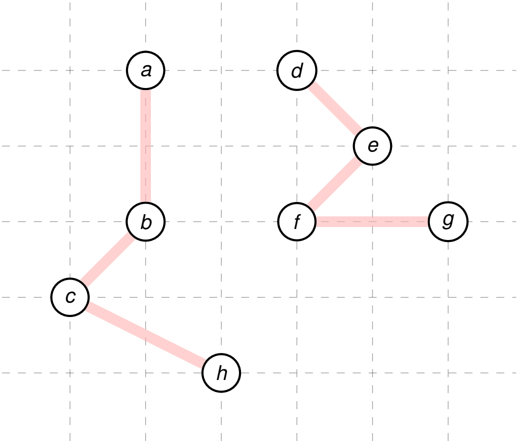

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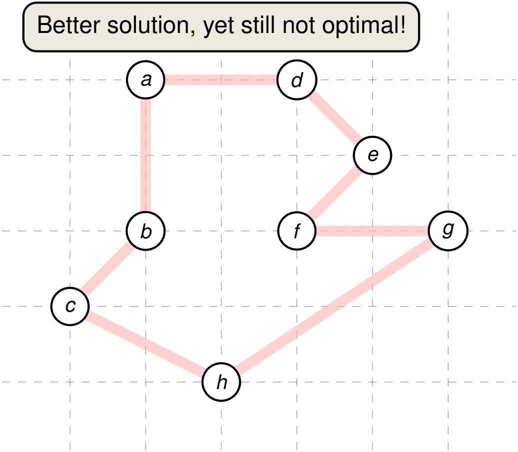

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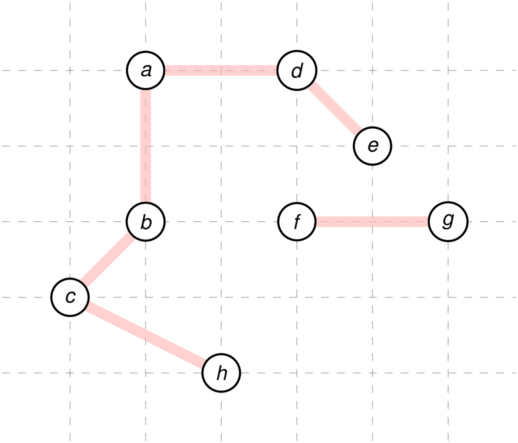
2. Perform preorder walk on MST T_{min} ✓
3. Return list of vertices according to the preorder tree walk ✓

Run of APPROX-TSP-TOUR



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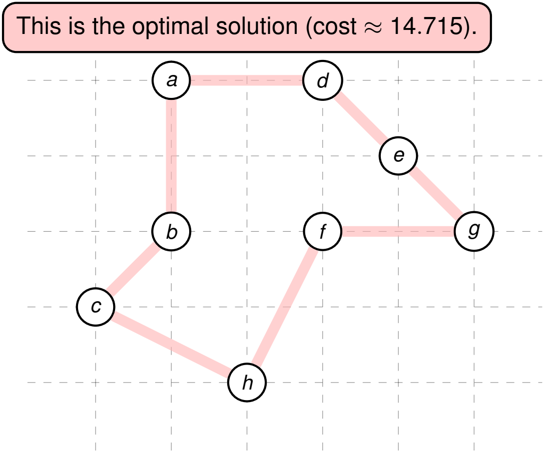

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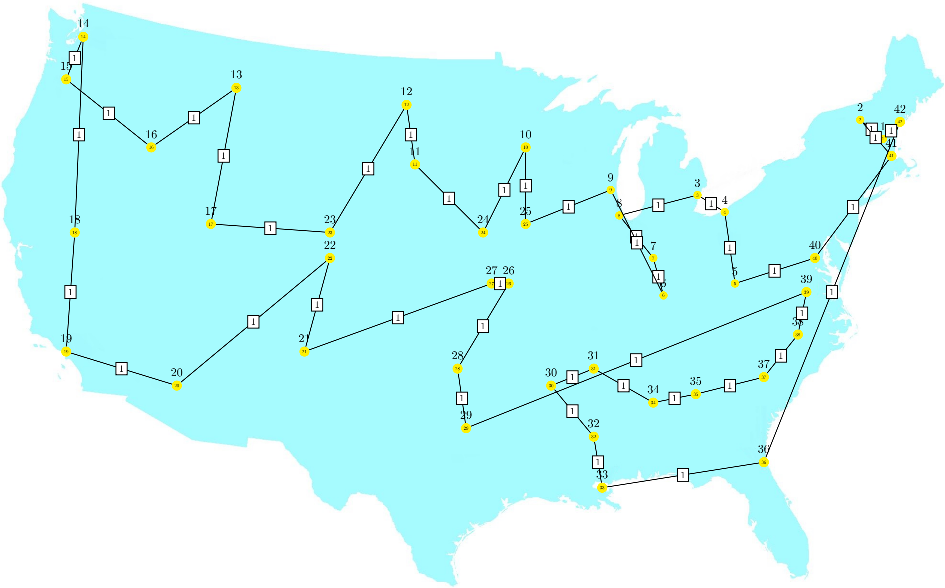
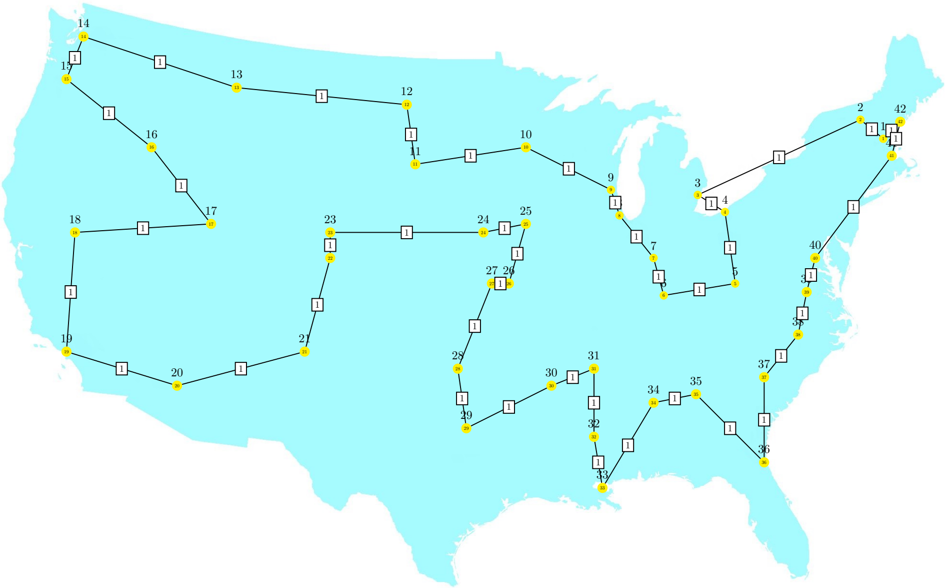
Approximate Solution: Objective 921



Optimal Solution: Objective 699



Proof of the Approximation Ratio

Theorem 35.2

APPROX-TSP-TOUR is a polynomial-time 2-approximation for the traveling-salesman problem with the triangle inequality.

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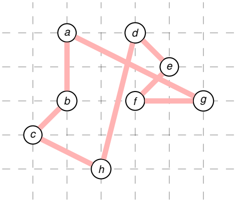

solution H of APPROX-TSP

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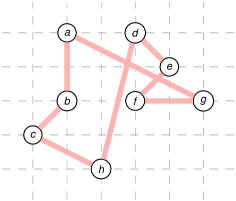

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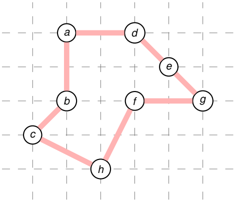

optimal solution H^*

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Proof:

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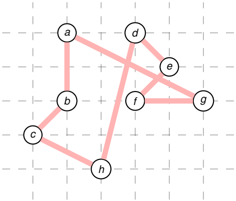

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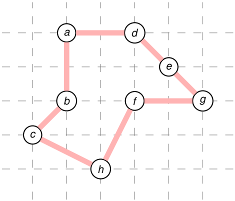

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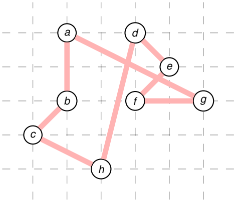

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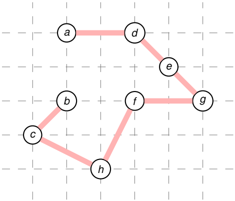

spanning tree T as a subset of H^*

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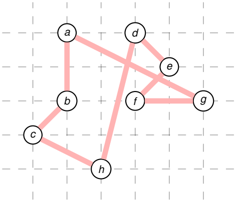

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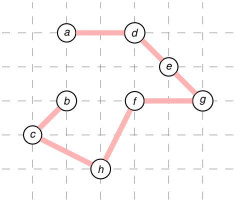

spanning tree T as a subset of H^*

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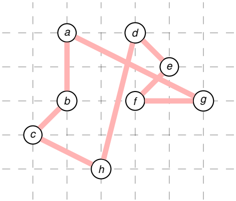
$\Rightarrow$ yields a spanning tree T and $c(T_{\min}) \leq c(T) \leq c(H^*)$



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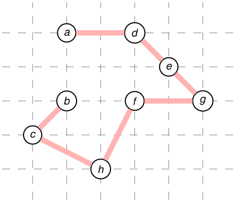

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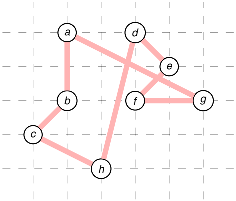
exploiting that all edge costs are non-negative!



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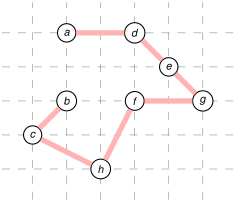

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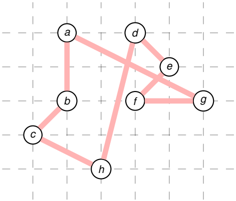
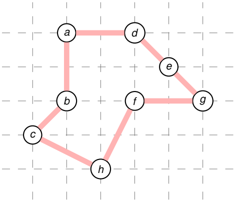


solution H of APPROX-TSP



optimal solution H^*

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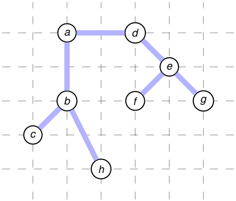

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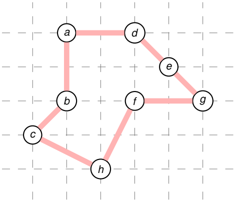

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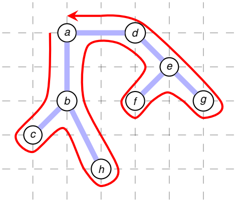
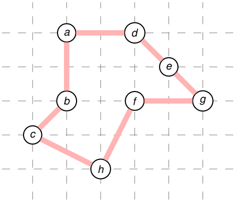


Walk $W = (a, b, c, b, h, b, a, d, e, f, e, g, e, d, a)$



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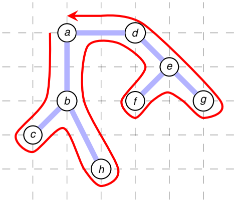

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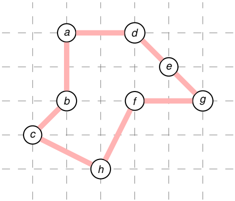

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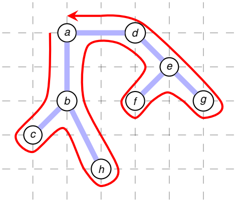

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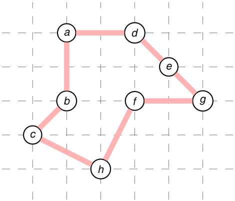

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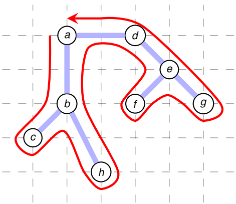

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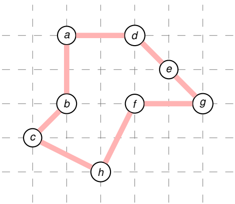

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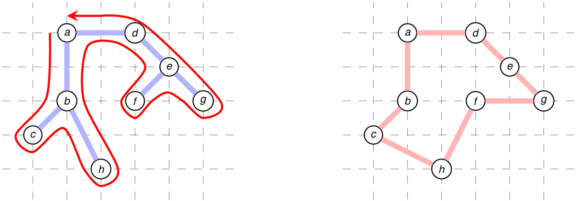
- Deleting duplicate vertices from W yields a tour H



Walk $W = (a, b, c, b, h, b, a, d, e, f, e, g, e, d, a)$ optimal solution H^*

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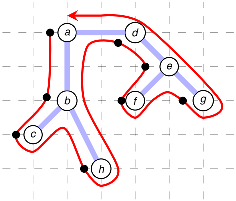

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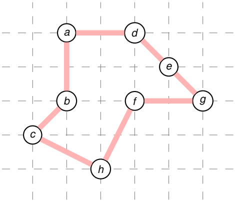

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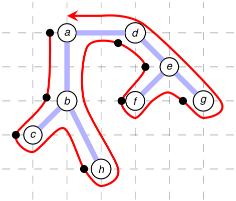
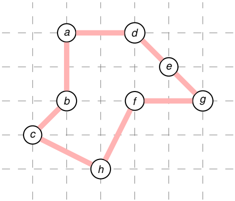


Walk $W = (a, b, c, \not b, h, \not b, \not a, d, e, f, \not e, g, \not e, \not d, a)$



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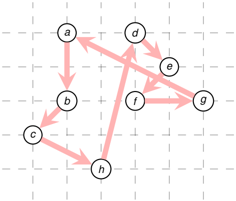
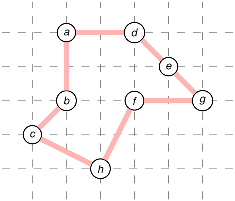


Tour $H = (a, b, c, h, d, e, f, g, a)$



optimal solution H^*

Proof of the Approximation Ratio

Theorem 35.2

APPROX-TSP-TOUR is a polynomial-time 2-approximation for the traveling-salesman problem with the triangle inequality.

Proof:

- Consider the optimal tour H^* and remove an arbitrary edge

$\Rightarrow$ yields a spanning tree T and $c(T_{\min}) \leq c(T) \leq c(H^*)$

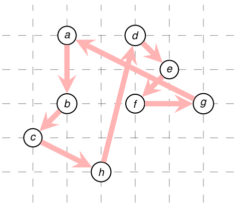
- Let W be the full walk of the minimum spanning tree $T_{\min}$ (including repeated visits)

$\Rightarrow$ Full walk traverses every edge exactly twice, so

$$c(W) = 2c(T_{\min}) \leq 2c(T) \leq 2c(H^*)$$

exploiting triangle inequality!

- Deleting duplicate vertices from W yields a tour H with smaller cost:



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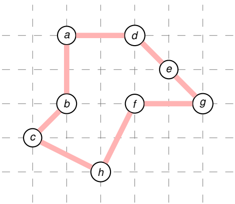

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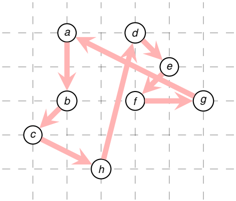

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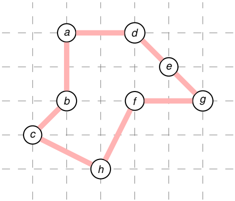

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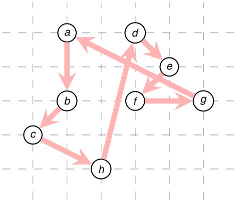

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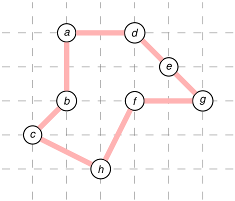

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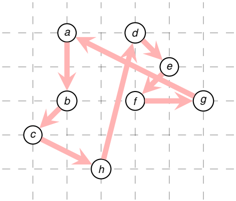

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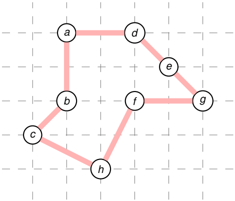

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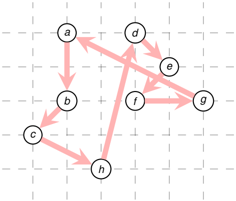

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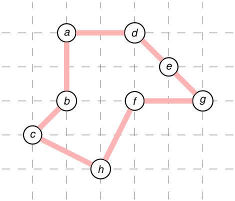

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Christofides Algorithm

Theorem 35.2

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Can we get a better approximation ratio?

Christofides Algorithm

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Can we get a better approximation ratio?

CHRISTOFIDES(G, c)
select a vertex $r \in G.V$ to be a "root" vertex
compute a minimum spanning tree $T_{\min}$ for G from root r
    using MST-PRIM(G, c, r)
compute a perfect matching $M_{\min}$ with minimum weight in the complete graph
    over the odd-degree vertices in $T_{\min}$
let H be a list of vertices, ordered according to when they are first visited
    in a Eulearian circuit of $T_{\min} \cup M_{\min}$
return the hamiltonian cycle H

Christofides Algorithm

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Theorem (Christofides'76)

There is a polynomial-time $\frac{3}{2}$-approximation algorithm for the travelling salesman problem with the triangle inequality.