Introduction to Probability

Lecture 11: Estimators (Part II)

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Outline

Recap: Unbiased Estimators and Bias

Definition

An estimator T is called an unbiased estimator for a parameter θ if

$$\mathbf{E}[T] = \theta,$$

irrespective of the value θ. The bias is defined as

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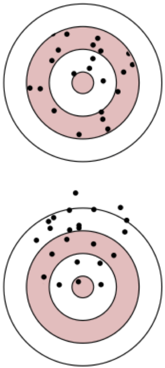

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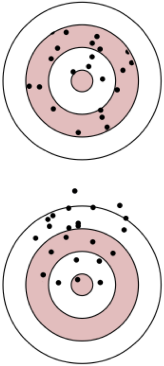

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- How can we measure the accuracy of an estimator?
  $\rightsquigarrow$ bias and mean-squared error
- If there are several unbiased estimators, which one to choose? $\rightsquigarrow$ mean-squared error (or variance)

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Estimating Population Sizes (First Version)

- Suppose we have a sample of a few serial numbers (IDs) of some product
- We assume IDs are running from 1 to an unknown parameter N (so $N = \theta$)
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  - their number must satisfy $n \leq N$

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- Achieving unbiasedness alone is not a good strategy
- Improvement: find an estimator which always returns a value at least $\max(X_1, X_2, \ldots, X_n)$

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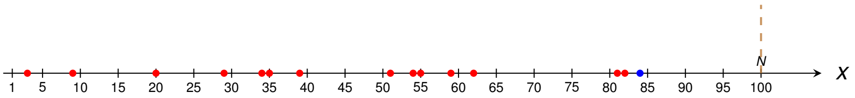

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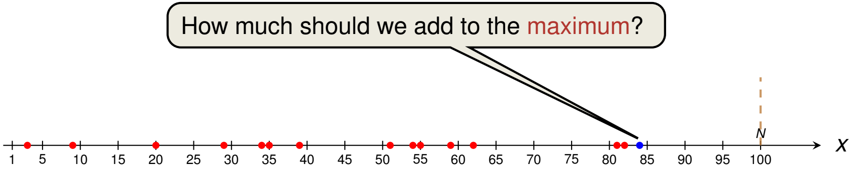

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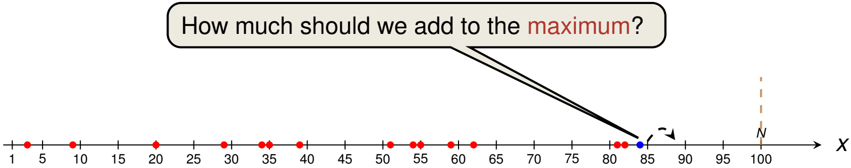

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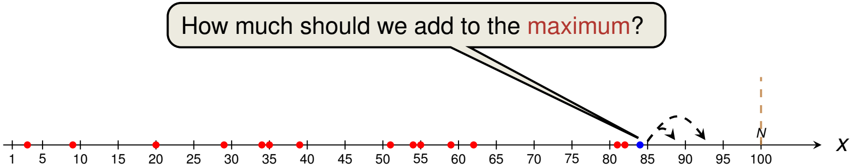

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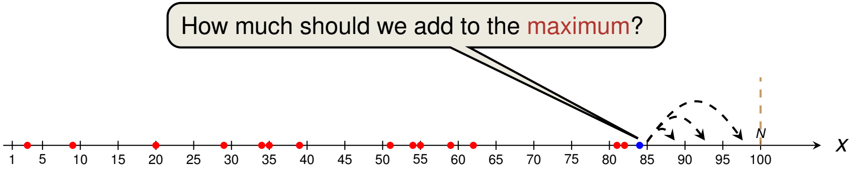

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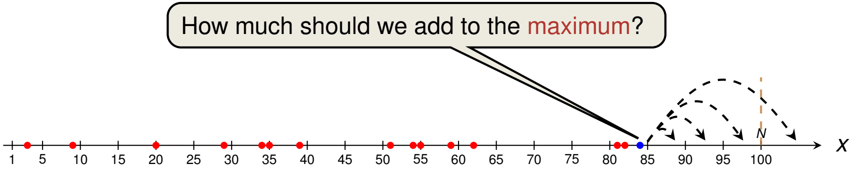

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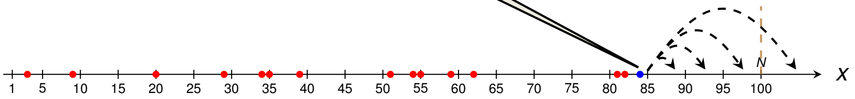

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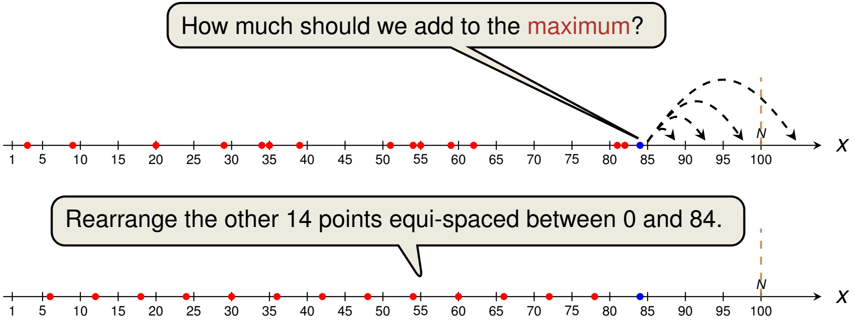

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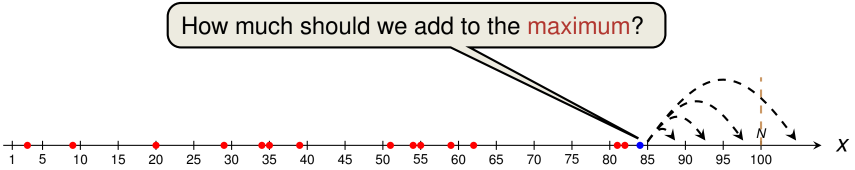

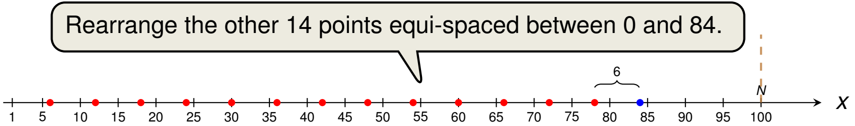

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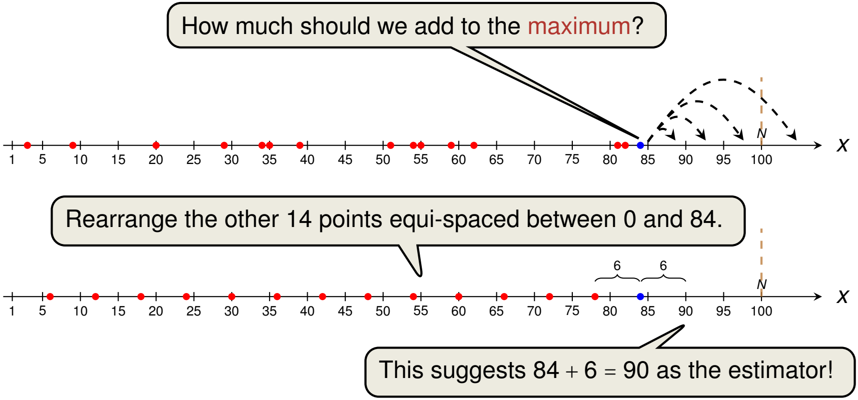

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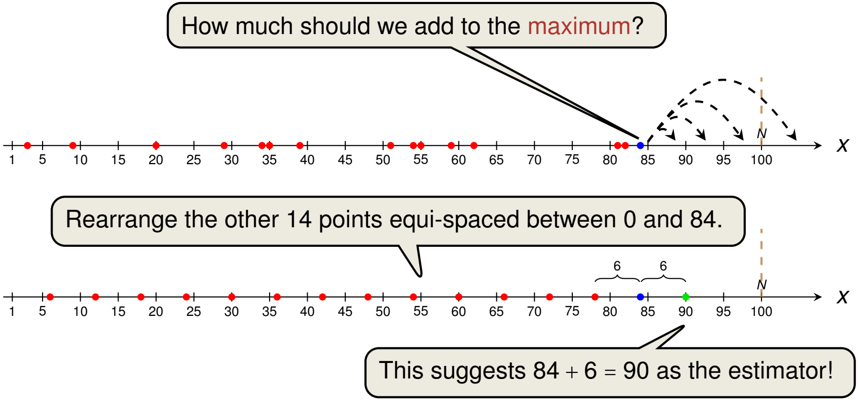

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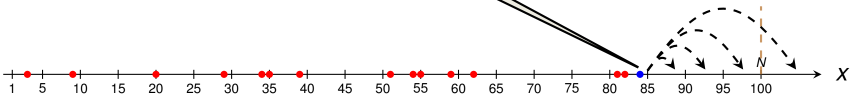

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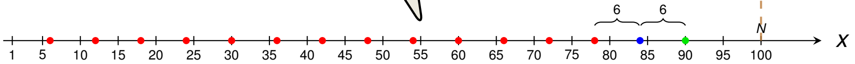

$\max(X_1, \ldots, X_n) + \frac{\max(X_1,\ldots,X_n)}{n-1}$

This suggests $84 + 6 = 90$ as the estimator!

Deriving the Estimator Based on Maximum

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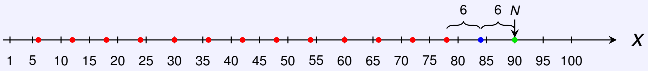

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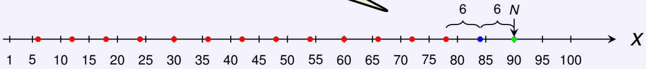

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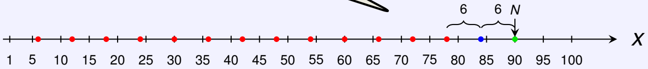

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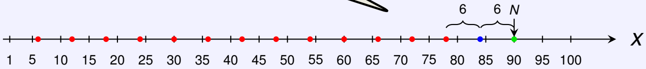

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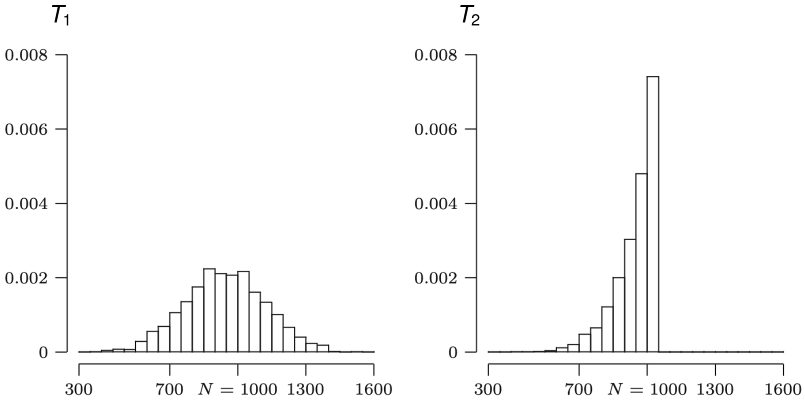
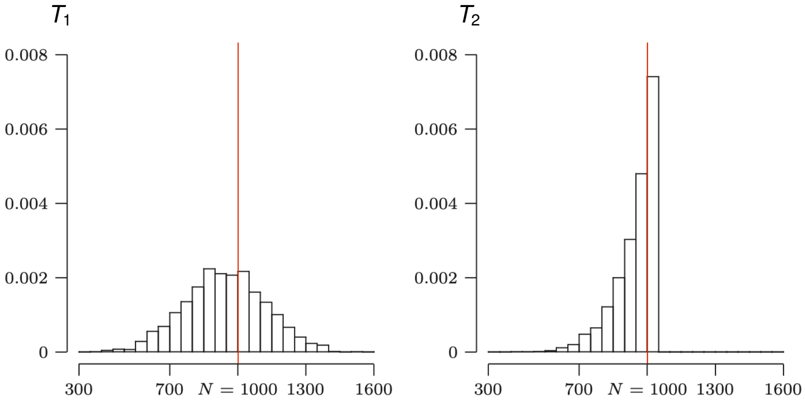


Source: Modern Introduction to Statistics

Figure: Histogram of 2000 values for T_1 and T_2, when $N = 1000$ and $n = 10$.

Empirical Analysis of the two Estimators



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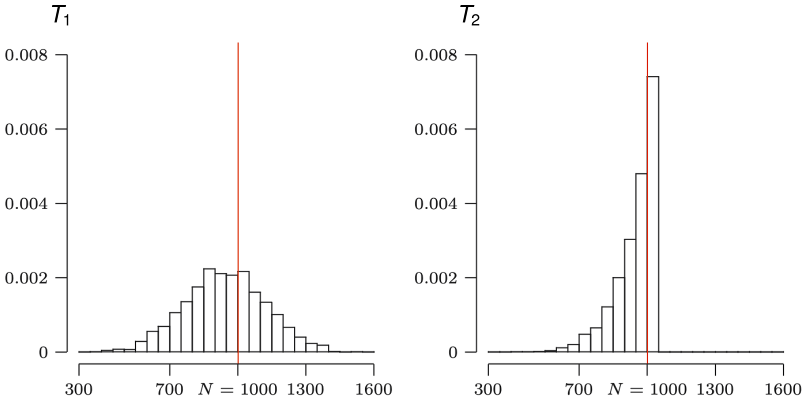

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Can we find a quantity that captures the superiority of T_2 over T_1?

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The mean squared error can be decomposed into:

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$\rightsquigarrow$ Minimum-Variance Unbiased Estimator (MVUE) (the unbiased estimator with the smallest variance).

Bias-Variance Decomposition: Derivation

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We need to prove: $\mathbf{MSE}[T] = (\mathbf{E}[T] - \theta)^2 + \mathbf{V}[T]$.

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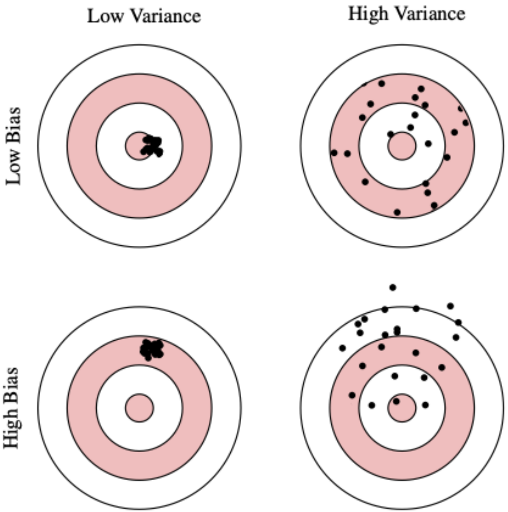

Source: Edwin Leuven (Point Estimation)

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- Rearranging and simplifying gives

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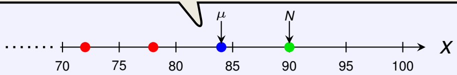

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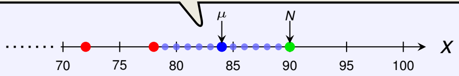

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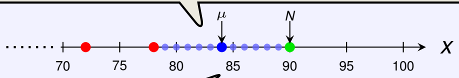

Maximum could have equally likely taken any value between 79 and 90

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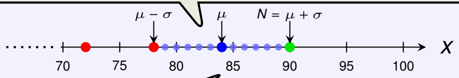

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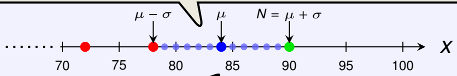

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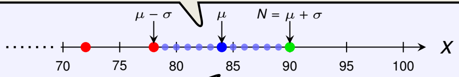

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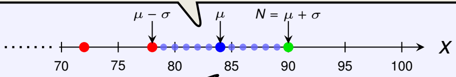

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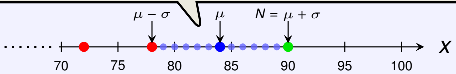

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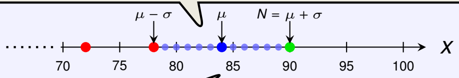

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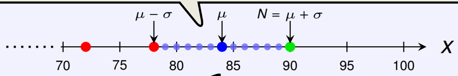

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- can be shown T_2 is the best unbiased estimator, i.e., it minimises MSE.

Outline

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Estimating Population Sizes

Mean Squared Error

Estimating Population Sizes through Collisions

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- Suppose $n = 6$, $N = 11$, $S = \{3, 4, 7, 8, 10, 15.83356, 20, 21, 56, 81, 10000\}$

A New Estimation Problem

Previous Model

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- Population/ID space of size $|S| = N$
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This also applies to situations where elements are not labelled before we see them first time (e.g., Mark & Recapture Method)

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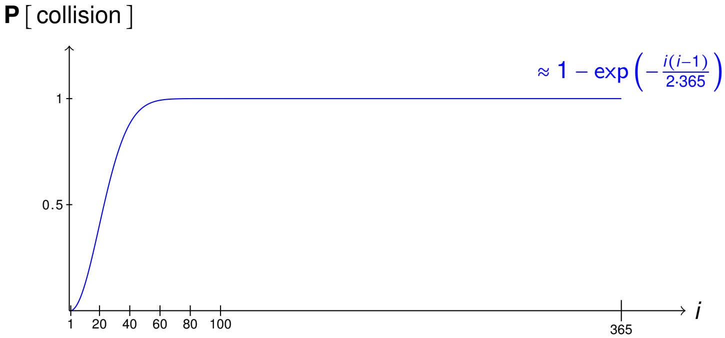

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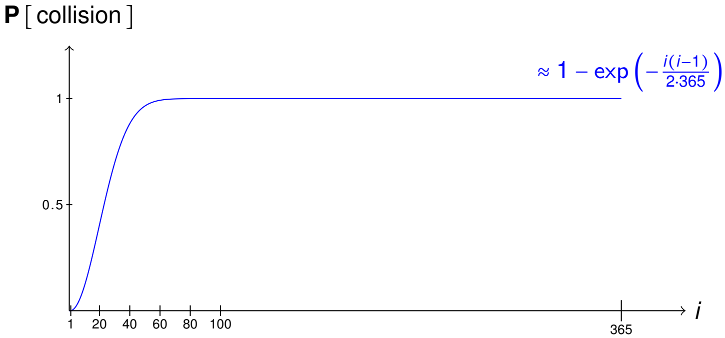

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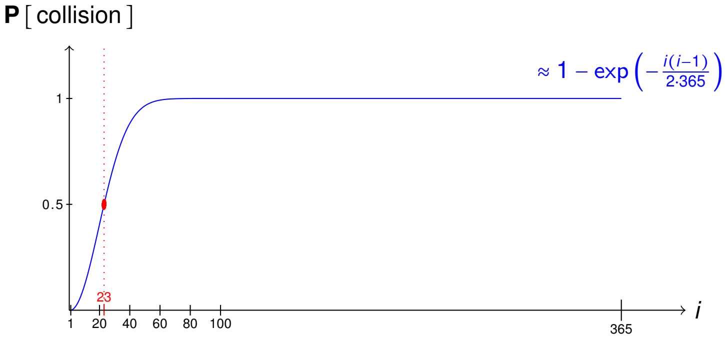

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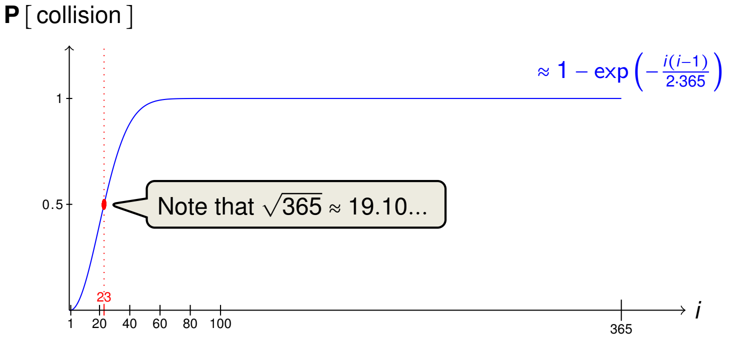

Estimation via Collision: The Algorithm

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FIND-FIRST-COLLISION(S)
C = ∅
For i = 1, 2, ...
    Take next i.i.d. sample X_i from S
    If X_i ∉ C then C ← C ∪ {X_i}
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Exercise: Prove a bound of $\leq 2 \cdot \sqrt{N}$

Estimation via Collision: Getting the Estimator Unbiased

Example 6

It is possible to define $T(i)$, $i \in \mathbb{N}$, such that $\mathbf{E}[\,T\,] = |S|$ for any set S.

Answer

Estimation via Collision: Getting the Estimator Unbiased

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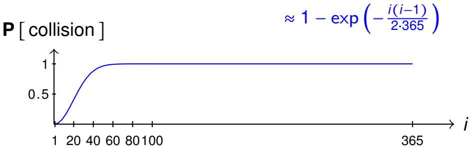
(proof that $T(i) = \binom{i}{2}$ is harder)

Mark & Recapture Method (non-examinable)



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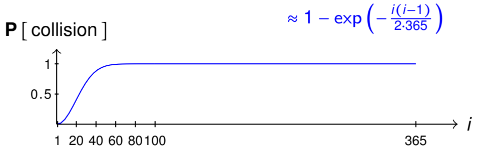

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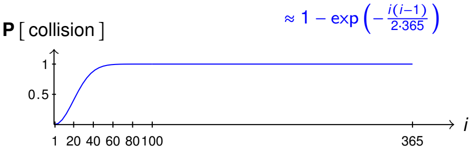

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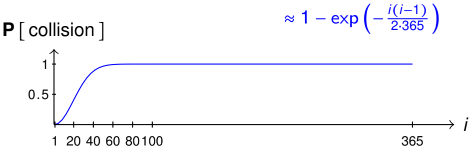

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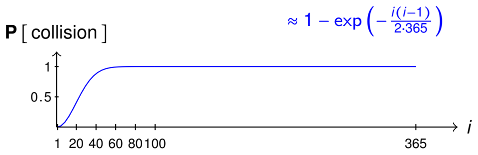

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$$\frac{k}{K} \approx \frac{n}{N} \qquad \Rightarrow \qquad N \approx n \cdot \frac{K}{k}.$$