

Introduction to Probability

Lecture 10: Estimators (Part I)

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Easter 2023



Announcements

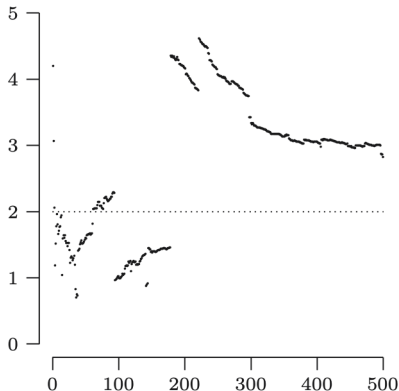
- No in-person Lectures 11, 12 (scheduled 22 May and 24 May)
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- possibly an in-person Example Class in the week 29 May–2 June

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- IA Examination Briefing on Wednesday 24 May 12:00-13:00 by Prof Robert Watson, Lecture Theatre A, Arts School (this venue!)
- for exam questions in this course, calculators are not required

A Distribution whose Average does not converge (Lecture 9)



$\text{Cau}(2, 1)$ distribution, Source: Modern Introduction to Statistics

The **Cauchy distribution** has “too heavy” tails (no expectation), in particular the average does not converge.

Outline

Introduction

Defining and Analysing Estimators

More Examples

Setting: We can take **random samples** in the form of **i.i.d. random variables** X_1, X_2, \dots, X_n from an **unknown distribution**.


- Taking enough samples allows us to estimate the **mean** (WLLN, CLT)
- Using indicator variables, we can estimate $\mathbf{P}[X \leq a]$ for any $a \in \mathbb{R}$
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Measurement = Quantity of Interest + Measurement Error

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Empirical Distribution Function

Definition of Empirical Distribution Function (Empirical CDF)

Let X_1, X_2, \dots, X_n being i.i.d. samples, and F be the corresponding distribution function. For any $a \in \mathbb{R}$, define

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Thus by taking enough samples, we can estimate the entire distribution (including its expectation and variance).

Empirical Distribution Functions (Example 1/2)

Example 1

Consider throwing an unbiased dice 8 times, and let the **realisation** be:

$$(x_1, x_2, \dots, x_8) = (4, 1, 5, 3, 1, 6, 4, 1).$$

What is the Empirical Distribution Function $F_8(a)$?

Answer

Empirical Distribution Functions (Example 1/2)

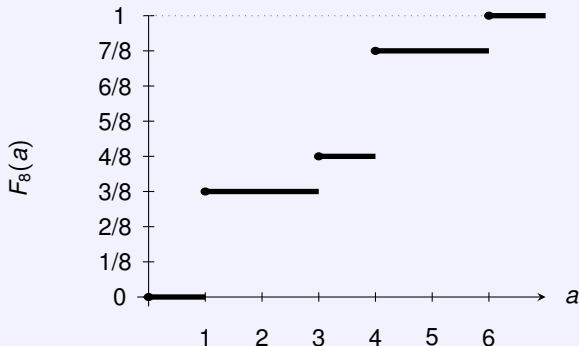
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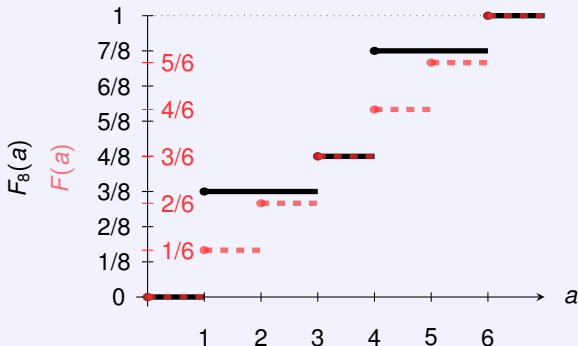
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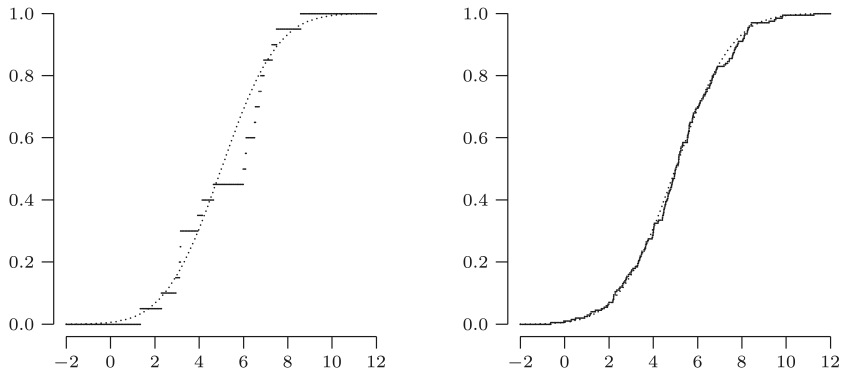
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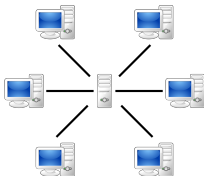
Source: Modern Introduction to Statistics

Figure: Empirical Distribution Functions of samples from a Normal Distribution $\mathcal{N}(5, 4)$ ($n = 20$ left, $n = 200$ right)

An Example of an Estimation Problem

Scenario

Consider the packages arriving at a network server.



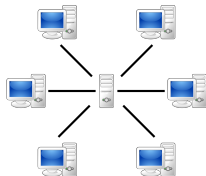
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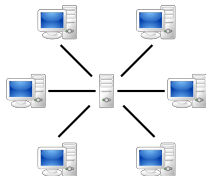
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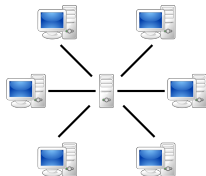
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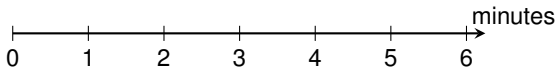
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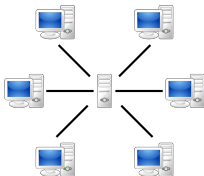


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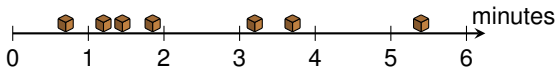
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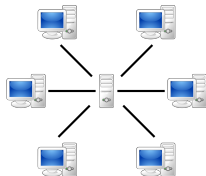


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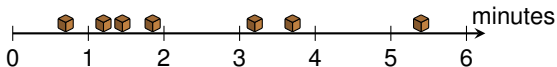
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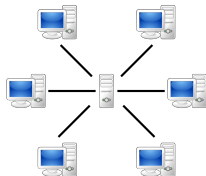


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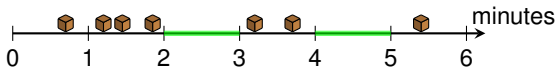
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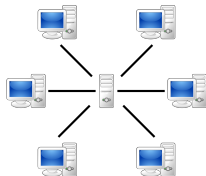


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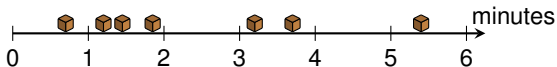
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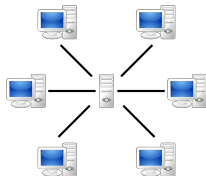


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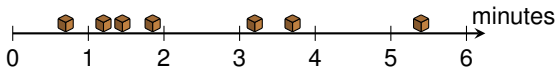
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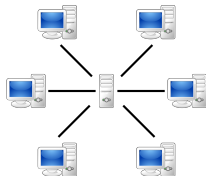


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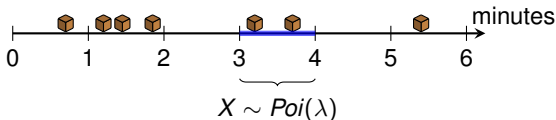
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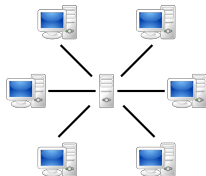


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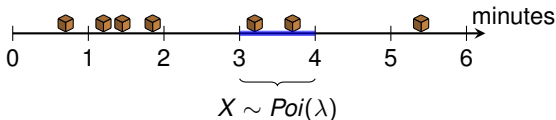
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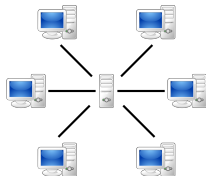
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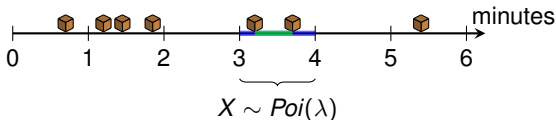
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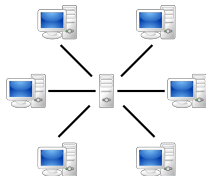
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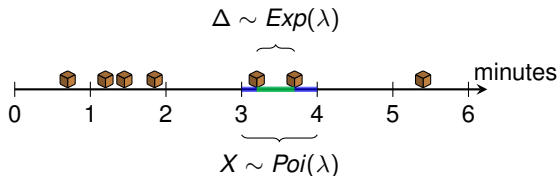
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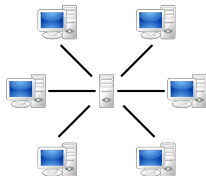
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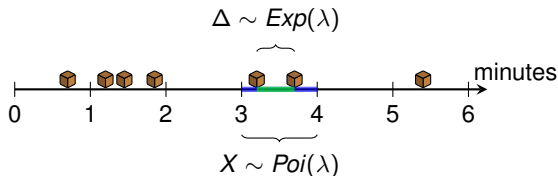
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Waiting Time (Lecture 5, Slide 22)



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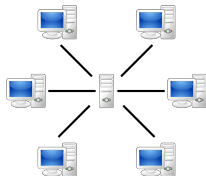
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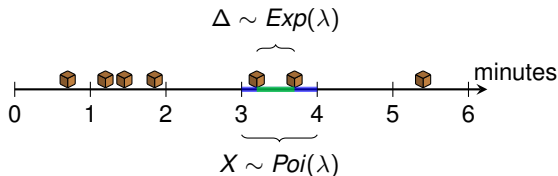
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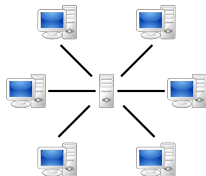
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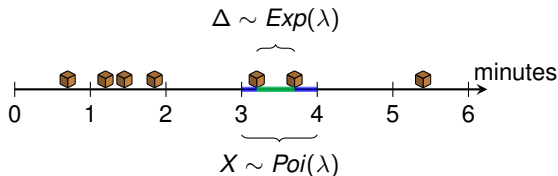
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Estimator for $e^{-\lambda}$

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A random variable

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Questions:

- What makes an **estimator** suitable? \rightsquigarrow **unbiased** (later: MSE)
- Does an **unbiased estimator** always exist? How to compute it?
- If there are several **unbiased** estimators, which one to choose?

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Example: Arrival of Packets (1/3)

- **Samples:** Given X_1, X_2, \dots, X_n i.i.d., $X_i \sim \text{Pois}(\lambda)$
- **Meaning:** X_i is the number of packets arriving in minute i



Example 2

Suppose we wish to estimate λ by using the sample mean \bar{X}_n .

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Applying the **Weak Law of Large Numbers**:

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Example: Arrival of Packets (2/3)

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Then

$$\mathbf{E}[Y_i] = \mathbf{P}[X_i = 0] = e^{-\lambda},$$

and thus we can define an estimator by

$$h_1(X_1, X_2, \dots, X_n) := \frac{Y_1 + Y_2 + \dots + Y_n}{n}.$$

Example: Arrival of Packets (3/3)

Example 3b

Suppose we wish to estimate the probability of zero arrivals $e^{-\lambda}$ by using the sample mean \bar{X}_n .

_____ Answer _____

Example: Arrival of Packets (3/3)

Example 3b

Suppose we wish to estimate the probability of zero arrivals $e^{-\lambda}$ by using the sample mean \bar{X}_n .

Answer

We saw that $\bar{X}_n = \frac{\sum_{i=1}^n X_i}{n}$ satisfies $\mathbf{E}[\bar{X}_n] = \mathbf{E}[X_1] = \lambda$.

Recall by the **Weak Law of Large Numbers**:

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Recall by the **Weak Law of Large Numbers**:

$$\lim_{n \rightarrow \infty} \mathbf{P} \left[\left| \bar{X}_n - \lambda \right| > \epsilon \right] = 0 \quad \text{for any } \epsilon > 0.$$

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Then we estimate $e^{-\lambda}$ by $e^{-\bar{X}_n}$. Hence our estimator is

$$h_2(X_1, X_2, \dots, X_n) := e^{-\bar{X}_n}.$$

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- Consider the **two estimators** $h_1(X_1, \dots, X_n)$ and $h_2(X_1, \dots, X_n)$.

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Behaviour of the Estimators

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For most values of λ , both estimators will never return the **exact** value of $e^{-\lambda}$ on the basis of 30 observations.

Simulation of the two Estimators

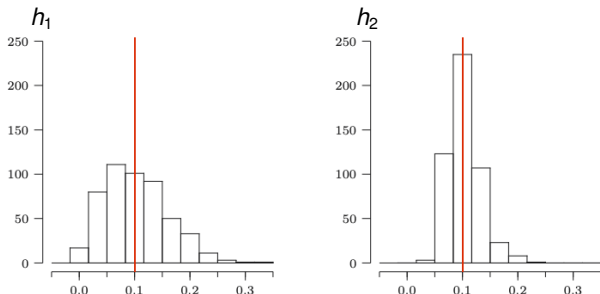
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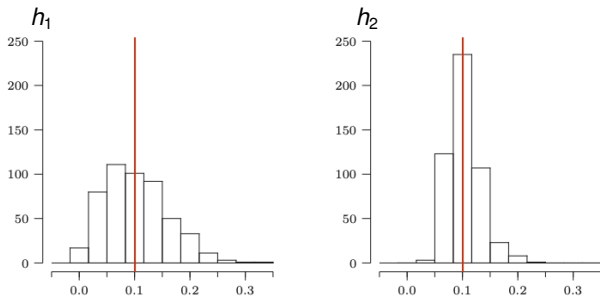
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Both estimators concentrate around the true value 0.1, but the second estimator appears to be more concentrated.

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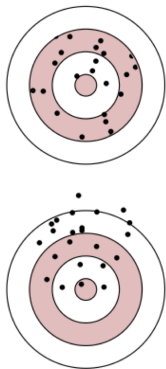
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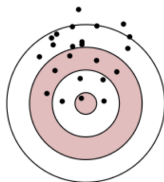
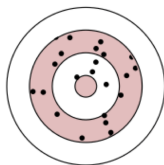
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Which of the two estimators h_1, h_2 are unbiased?



Example 4a

Is $h_1(X_1, X_2, \dots, X_n) = \frac{Y_1 + Y_2 + \dots + Y_n}{n}$ an unbiased estimator for $e^{-\lambda}$?

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Bias of the Second Estimator (and Jensen's Inequality)

Example 4b

Is $h_2(X_1, X_2, \dots, X_n) = e^{-\bar{X}_n}$ an unbiased estimator for $e^{-\lambda}$?

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Bias of the Second Estimator (and Jensen's Inequality)

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Is $h_2(X_1, X_2, \dots, X_n) = e^{-\bar{X}_n}$ an unbiased estimator for $e^{-\lambda}$?

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For any random variable X , and any **convex function** $g : \mathbb{R} \rightarrow \mathbb{R}$, we have

$$\mathbf{E}[g(X)] \geq g(\mathbf{E}[X]).$$

If g is **strictly convex** and X is not constant, then the inequality is strict.

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- This follows by **Jensen's inequality**, and the inequality is **strict** since $z \mapsto e^{-z}$ is **strictly convex**.
- Thus $h_2(X_1, X_2, \dots, X_n)$ is not unbiased – it has **positive bias**.

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Asymptotic Bias of the Second Estimator (non-examinable)

Example 4c

$\mathbf{E}[h_2(X_1, \dots, X_n)] \xrightarrow{n \rightarrow \infty} e^{-\lambda}$ (hence it is **asymptotically unbiased**).

Answer

- Recall $h_2(X_1, \dots, X_n) = e^{-\bar{X}_n}$. For any $0 \leq k \leq n$,

$$\mathbf{P}\left[h_2(X_1, \dots, X_n) = e^{-k/n}\right] = \mathbf{P}\left[\sum_{i=1}^n X_i = k\right] = \mathbf{P}[Z = k],$$

where $Z \sim \text{Pois}(n \cdot \lambda)$ (since $\text{Pois}(\lambda_1) + \text{Pois}(\lambda_2) = \text{Pois}(\lambda_1 + \lambda_2)$)

$$\Rightarrow \mathbf{P}\left[h_2(X_1, \dots, X_n) = e^{-k/n}\right] = \frac{e^{-n\lambda} \cdot (n\lambda)^k}{k!}$$

$$\begin{aligned} \Rightarrow \mathbf{E}[h_2(X_1, \dots, X_n)] &= \sum_{k=0}^{\infty} e^{-n\lambda} \cdot \frac{(n\lambda)^k}{k!} \cdot e^{-k/n} \\ &\stackrel{\text{By LOTUS}}{=} e^{-n\lambda} \cdot e^{n\lambda e^{-1/n}} \sum_{k=0}^{\infty} e^{-n\lambda e^{-1/n}} \cdot \frac{(n\lambda e^{-1/n})^k}{k!} \\ &= e^{-n\lambda \cdot (1 - e^{-1/n})} \cdot 1 \end{aligned}$$

since $e^x = 1 + x + O(x^2)$ for small x

$$\stackrel{n \rightarrow \infty}{\approx} e^{-n\lambda \cdot (1 - 1 + 1/n + O(1/n^2))} = e^{-\lambda + O(\lambda/n)}.$$

Hence in the limit, the positive bias of h_2 diminishes.

Outline

Introduction

Defining and Analysing Estimators

More Examples

Unbiased Estimators for Expectation and Variance

Let X_1, X_2, \dots, X_n be **identically distributed** samples from a distribution with finite expectation μ and finite variance σ^2 . Then

$$\bar{X}_n := \frac{X_1 + X_2 + \dots + X_n}{n}$$

is an **unbiased** estimator for μ .

Furthermore,

$$S_n = S_n(X_1, \dots, X_n) := \frac{1}{n-1} \cdot \sum_{i=1}^n (X_i - \bar{X}_n)^2$$

is an **unbiased** estimator for σ^2 .

Example 5

We need to prove: $\mathbf{E}[S_n] = \sigma^2$.

Answer

Multiplying by $n - 1$ yields:

$$\begin{aligned}(n-1) \cdot S_n &= \sum_{i=1}^n (X_i - \bar{X}_n)^2 \\&= \sum_{i=1}^n (X_i - \mu + \mu - \bar{X}_n)^2 \\&= \sum_{i=1}^n (X_i - \mu)^2 + \sum_{i=1}^n (\bar{X}_n - \mu)^2 - 2 \sum_{i=1}^n (X_i - \mu) (\bar{X}_n - \mu) \\&= \sum_{i=1}^n (X_i - \mu)^2 + n (\bar{X}_n - \mu)^2 - 2 (\bar{X}_n - \mu) \cdot n \cdot (\bar{X}_n - \mu) \\&= \sum_{i=1}^n (X_i - \mu)^2 - n (\bar{X}_n - \mu)^2.\end{aligned}$$

Let us now take **expectations**:

$$\begin{aligned}(n-1) \cdot \mathbf{E}[S_n] &= \sum_{i=1}^n \mathbf{E}[(X_i - \mu)^2] - n \cdot \mathbf{E}[(\bar{X}_n - \mu)^2] \\&= n \cdot \sigma^2 - n \cdot \sigma^2/n \\&= (n-1) \cdot \sigma^2.\end{aligned}$$

Recall: $\mathbf{E}[(\bar{X}_n - \mu)^2] = \mathbf{V}[\bar{X}_n] = \sigma^2/n$

Example 5 (cntd.)

$$\mathbf{E}[S_n] = \mathbf{E}\left[\frac{1}{n-1} \cdot \sum_{i=1}^n (X_i - \bar{X}_n)^2\right] = \sigma^2. \text{ Why is it } \frac{1}{n-1} \text{ and not } \frac{1}{n}?$$

Answer

- **First Explanation.** Consider $n = 1$. Having just one estimate should not tell us anything about the variance (it could be infinite!).
- **Second Explanation.** Assume μ is known, but σ^2 unknown. Define

$$\sum_{i=1}^n (X_i - \mu)^2 =: A.$$

- Additionally, define

$$\sum_{i=1}^n (X_i - \bar{X}_n)^2 =: B.$$

- $B \leq A$, as \bar{X}_n solves a quadratic minimisation problem.
- It is easy to verify that $\frac{1}{n} \cdot A$ is an unbiased estimator for σ^2
- The factor $\frac{1}{n-1}$ (instead of $\frac{1}{n}$) corrects the fact that \bar{X}_n is a more “favourable” average than the true mean λ .

Warning: An Unbiased Estimator may not always exist

Example 6

Suppose that we have one sample $X \sim \text{Bin}(n, p)$, where $0 < p < 1$ is unknown but n is known. Prove there is **no unbiased estimator** for $1/p$.

Answer

- First a simpler proof which exploits that p might be arbitrarily small
- **Intuition:** For very small p , one $T(k)$, $k \in \{0, 1, \dots, n\}$ must be very large, but then $\mathbf{E}[T(X)]$ is too large for, e.g., $p = 1/2$
- **Formal Argument:**
 - Assume $T(X)$ is an unbiased estimator for $\frac{1}{p}$ for all $0 < p < 1$
 - Define $M := \max_{0 \leq k \leq n} T(k)$. Then,

$$\begin{aligned}\mathbf{E}[T(X)] &= \sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k} \cdot T(k) \\ &\leq M \cdot \sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k} = M.\end{aligned}$$

- Hence this estimator does not work for $p < \frac{1}{M}$, since then $\mathbf{E}[T(X)] \leq M < \frac{1}{p}$ (negative bias!)
- The next proof will work even if $p \in [a, b]$ for $0 < a < b \leq 1$.

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Example 6 (cntd.)

Suppose that we have one sample $X \sim \text{Bin}(n, p)$, where $0 < p < 1$ is unknown but n is known. Prove there is **no unbiased estimator** for $1/p$.

Answer

- Suppose there exists an unbiased estimator with $\mathbf{E}[T(X)] = 1/p$.
- Then

$$\begin{aligned} 1 &= p \cdot \mathbf{E}[T(X)] \\ &= p \cdot \sum_{k=0}^n \mathbf{P}[X = k] \cdot T(k) \\ &= p \cdot \sum_{k=0}^n \binom{n}{k} p^k \cdot (1-p)^{n-k} \cdot T(k) \end{aligned}$$

- Last term is a **polynomial of degree $n+1$** with constant term zero
 $\Rightarrow p \cdot \mathbf{E}[T(X)] - 1$ is a **(non-zero) polynomial of degree $\leq n+1$**
 \Rightarrow this polynomial has at most $n+1$ roots
 $\Rightarrow \mathbf{E}[T(X)]$ can be equal to $1/p$ for at most $n+1$ values of p , and thus cannot be an unbiased.