Introduction to Probability

Lectures 9: Central Limit Theorem

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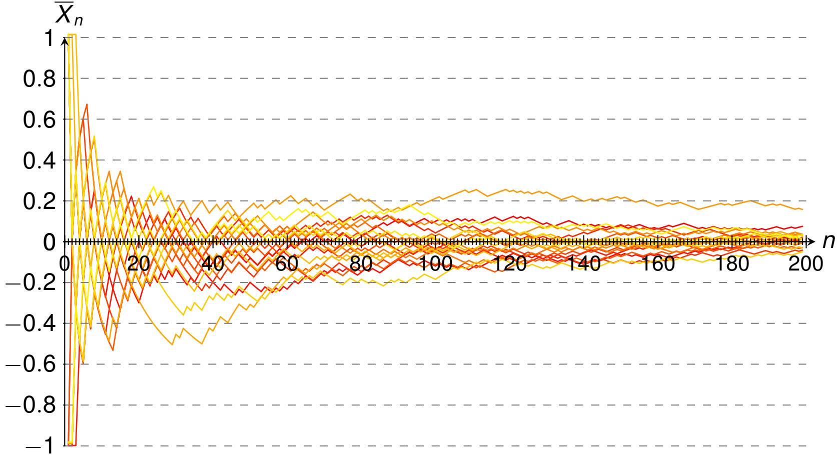
UNIVERSITY OF CAMBRIDGE

Outline

Weak Law of Large Numbers (4/4)

Weak Law of Large Numbers: For any $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbf{P}\left[\, |\overline{X}_n - \mu| > \epsilon \,\right] = 0$$



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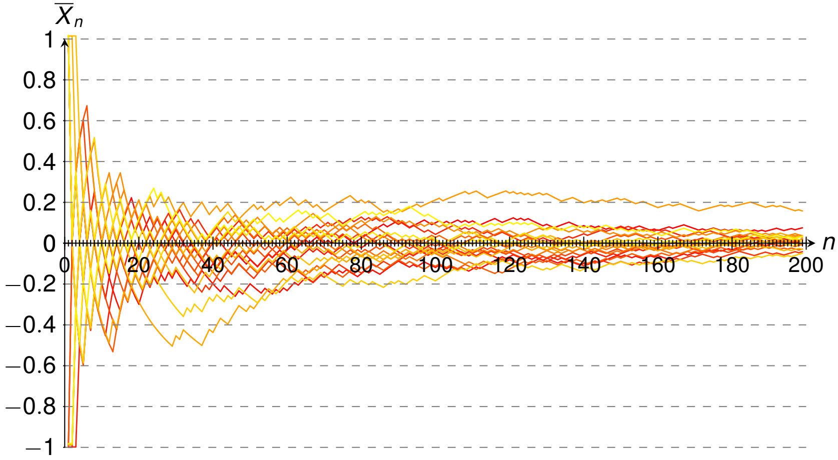

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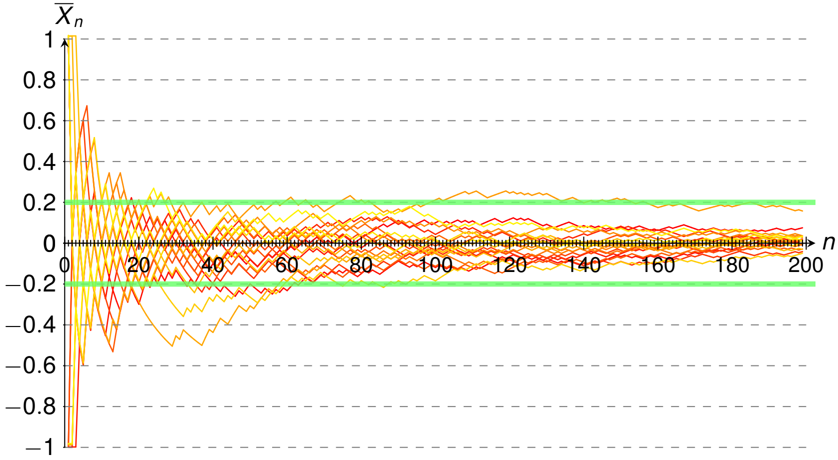

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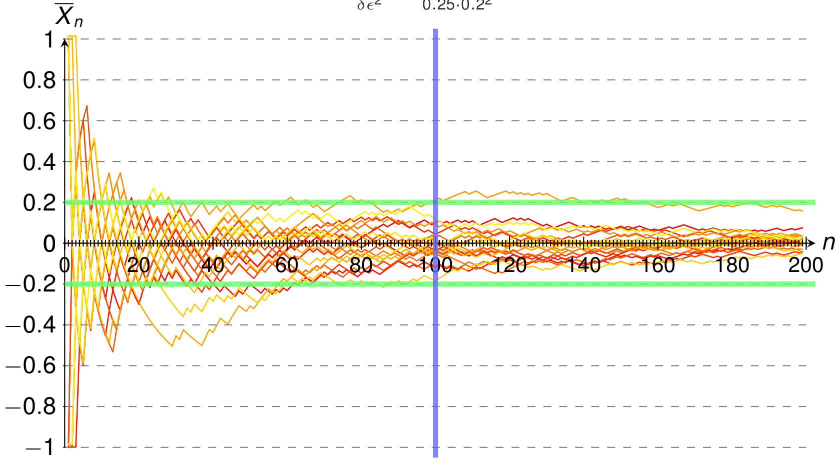

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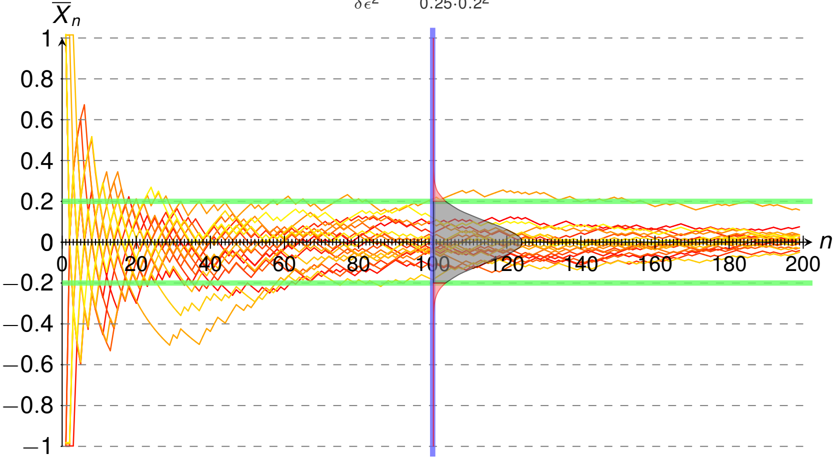

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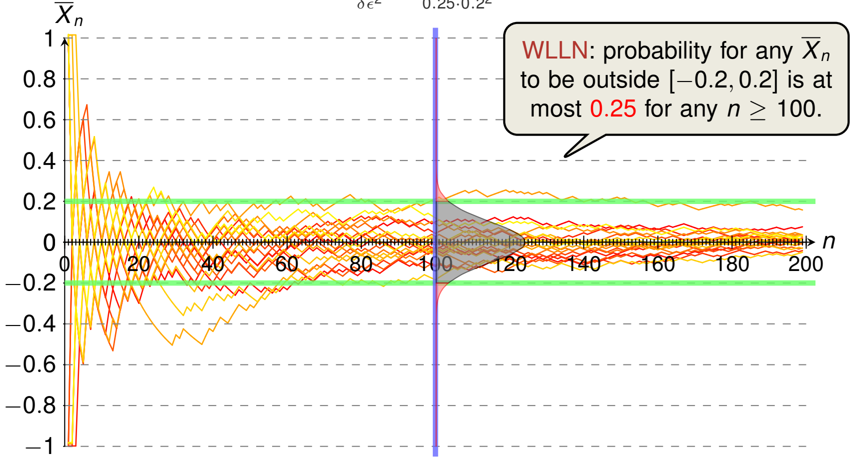

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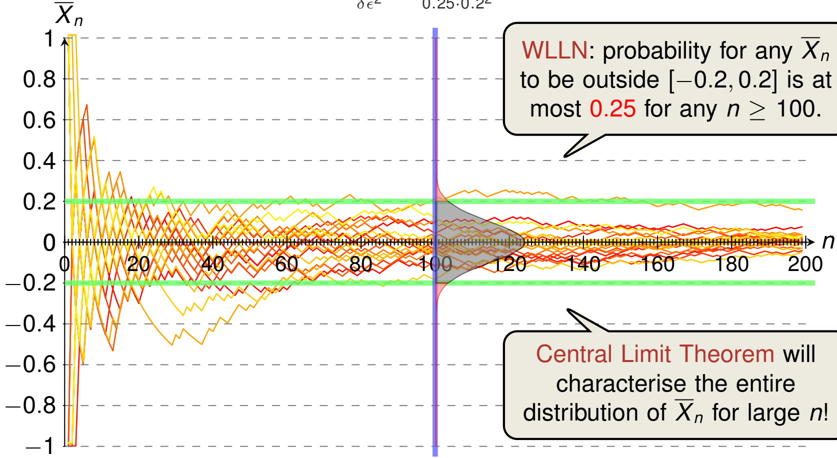

WLLN: probability for any $\overline{X}_n$ to be outside $[-0.2, 0.2]$ is at most 0.25 for any $n \geq 100$.

Central Limit Theorem will characterise the entire distribution of $\overline{X}_n$ for large n!

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Towards the CLT: Finding the Right Scaling

- Let $X_1, X_2, \ldots$ i.i.d. with $\mu = 0$ and finite σ^2

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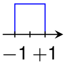

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The Sum

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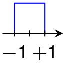
- The variance is $\mathbf{V}\left[\widetilde{X}_n\right] = n\sigma^2 \to \infty$



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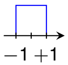

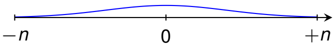

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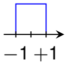

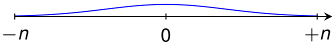

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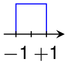

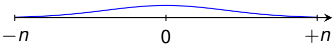

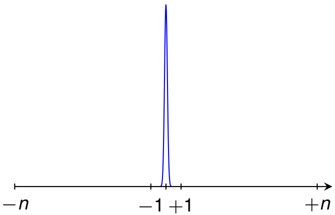

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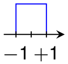

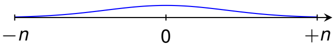

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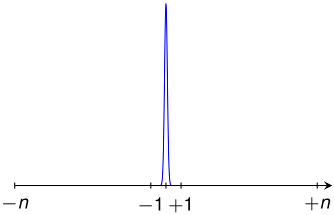

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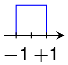

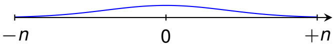

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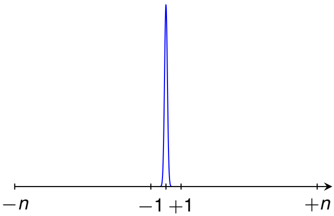

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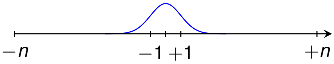

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Then for any number $a \in \mathbb{R}$, it holds that

$$\lim_{n \to \infty} F_{Z_n}(a) = \Phi(a) \qquad ,$$

where Φ is the distribution function of the $\mathcal{N}(0, 1)$ distribution.

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In words: the distribution of Z_n **always** converges to the distribution function Φ of the standard normal distribution.

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- one of the most remarkable results in probability/statistics
- extremely powerful tool in applications: we may not know the actual distribution in real-world, and CLT says we don't have to(!)
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- usually $n \geq 10$ or $n \geq 15$ is sufficient in practice
- approximation tends to be worse when threshold a is far from 0, distribution of X_i's asymmetric, bimodal or discrete

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Recap: Weak Law of Large Numbers

Central Limit Theorem

Illustrations

Examples

Bonus Material (non-examinable)

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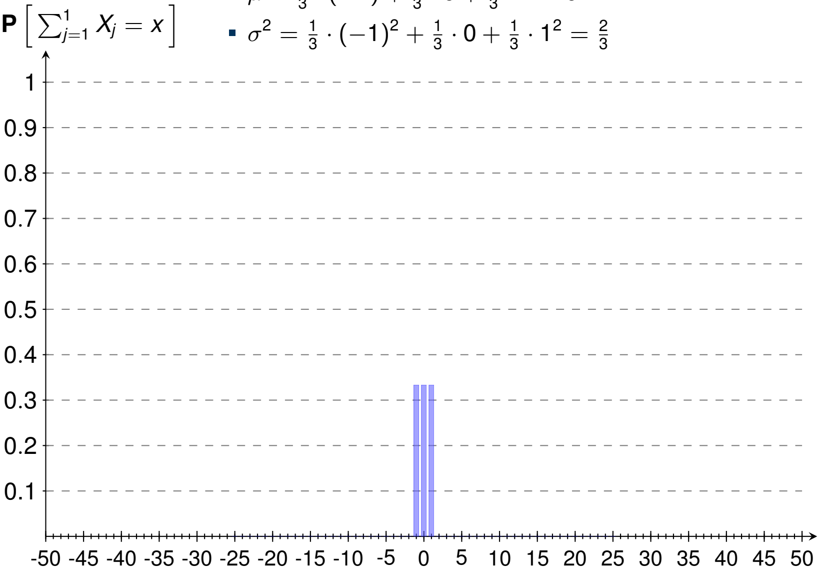

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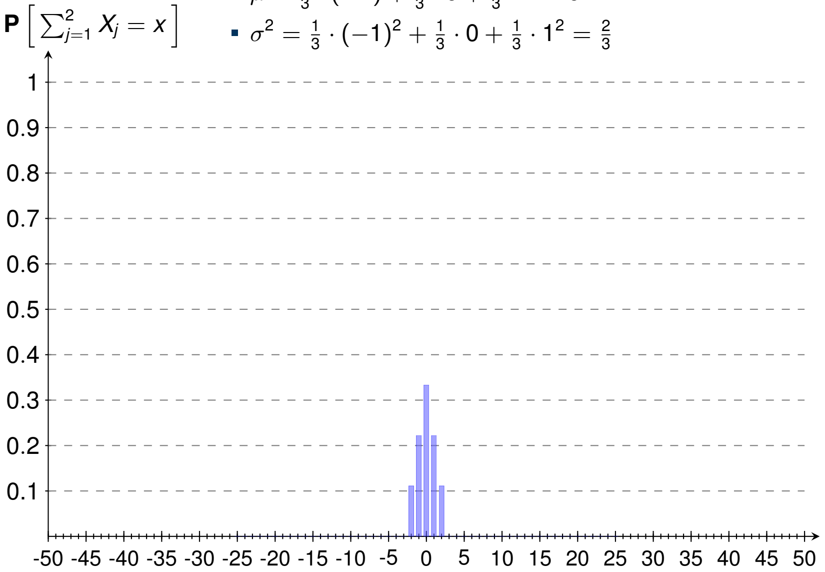

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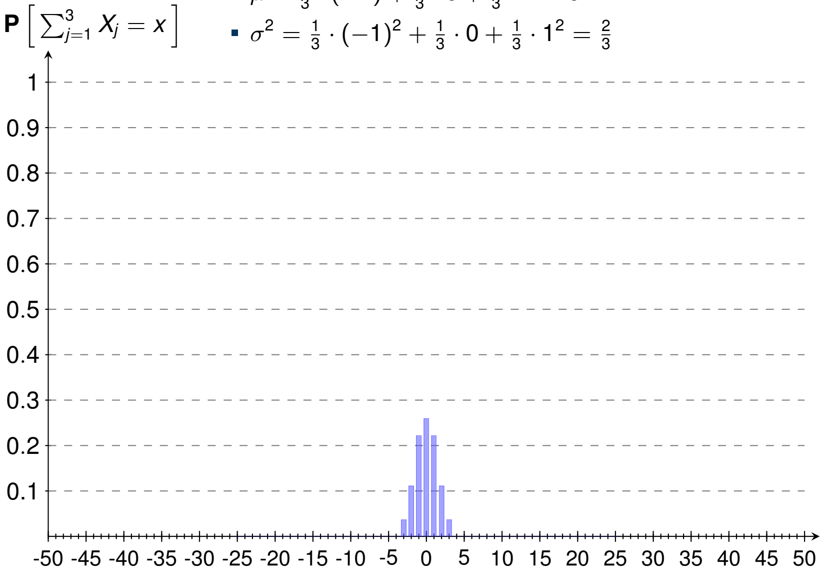

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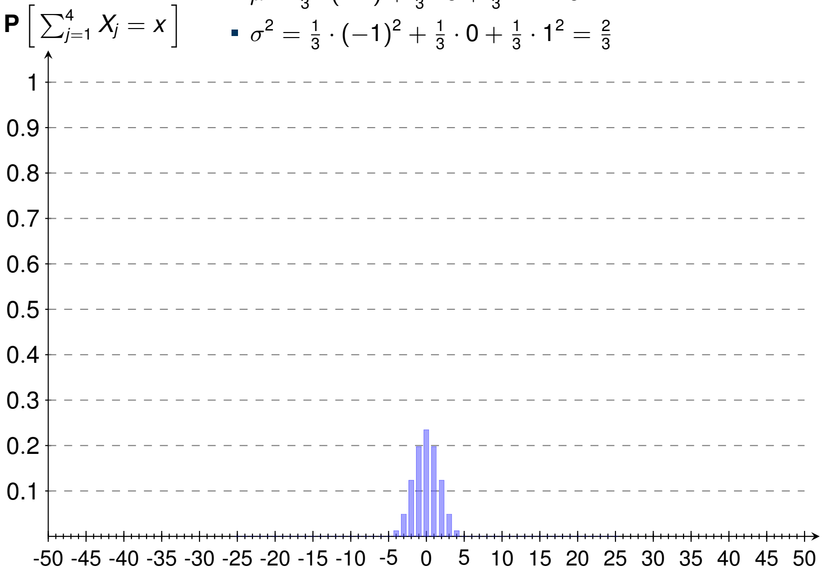

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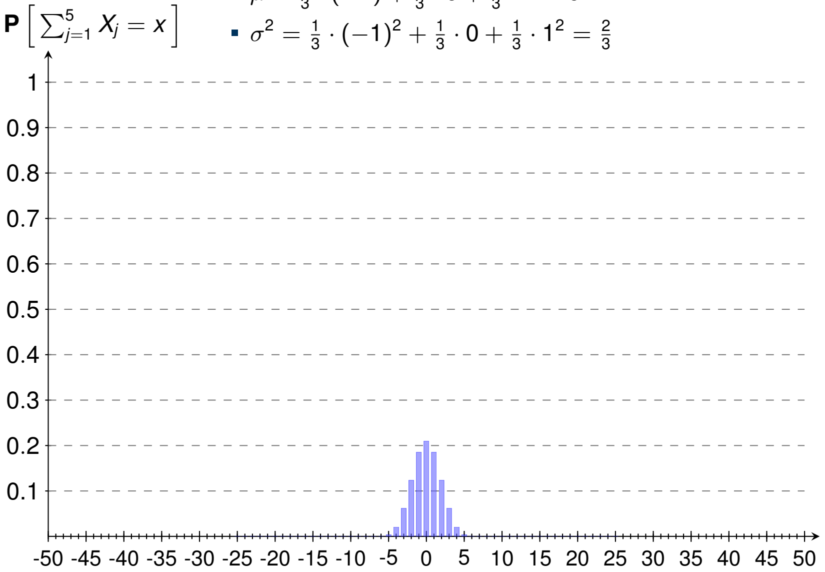

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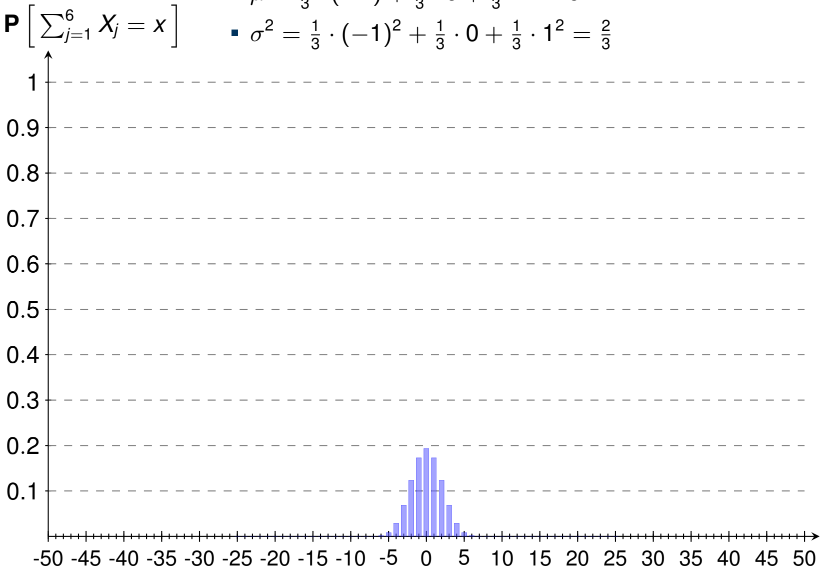

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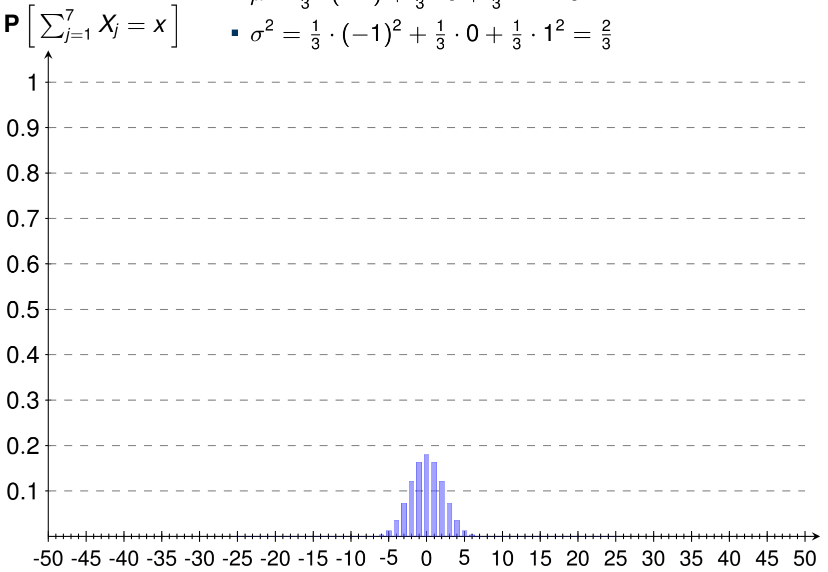

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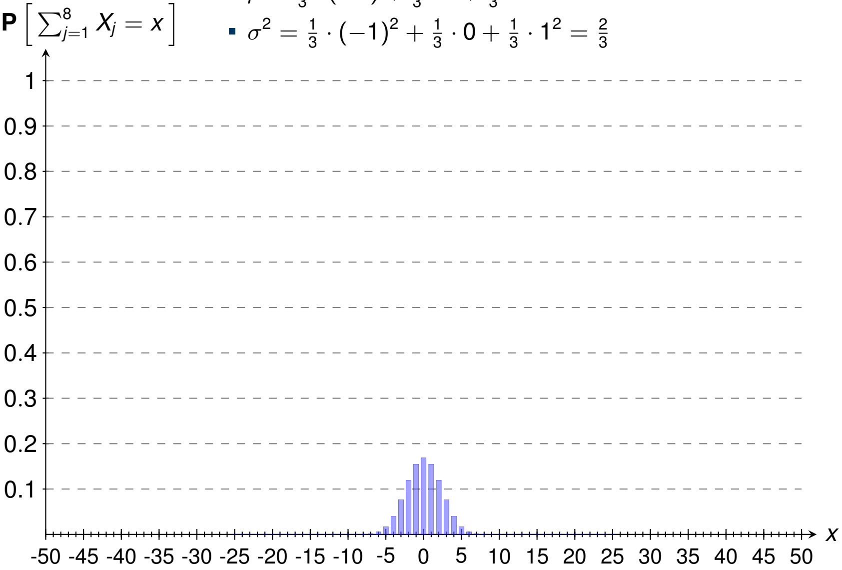

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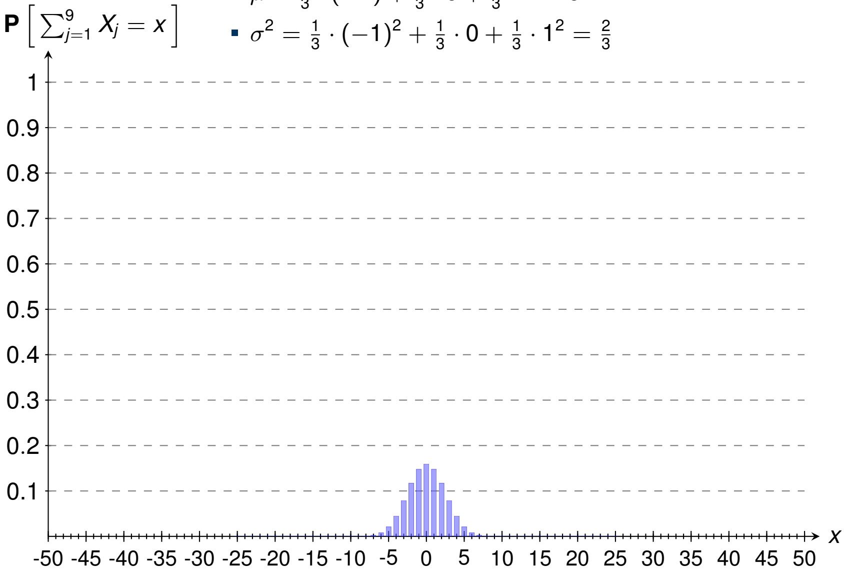

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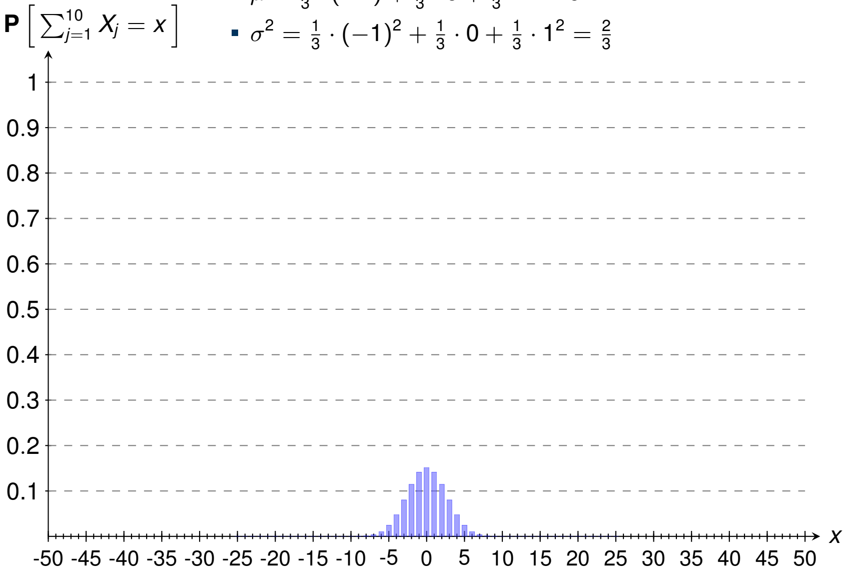

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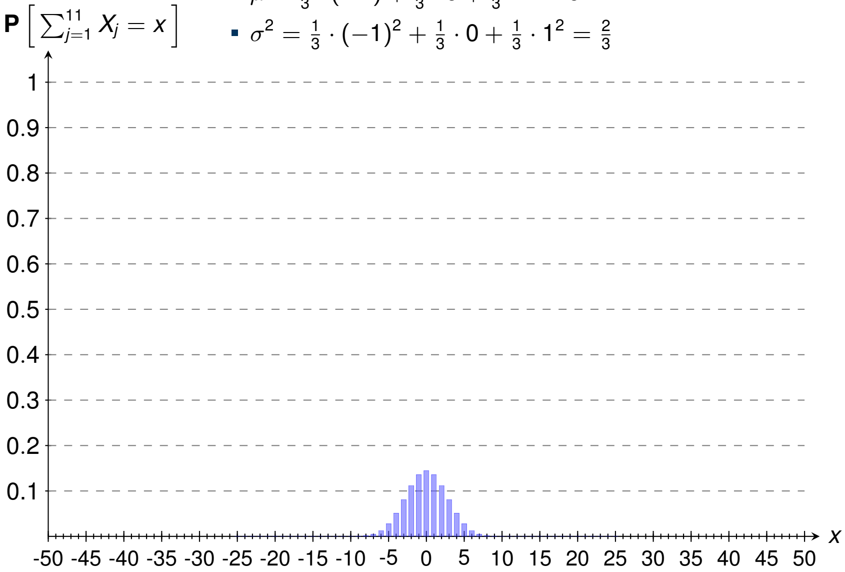

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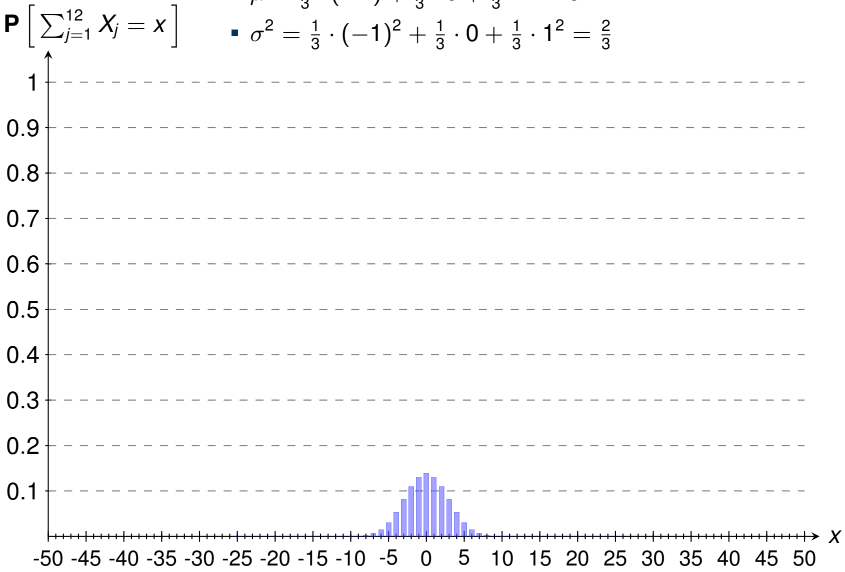

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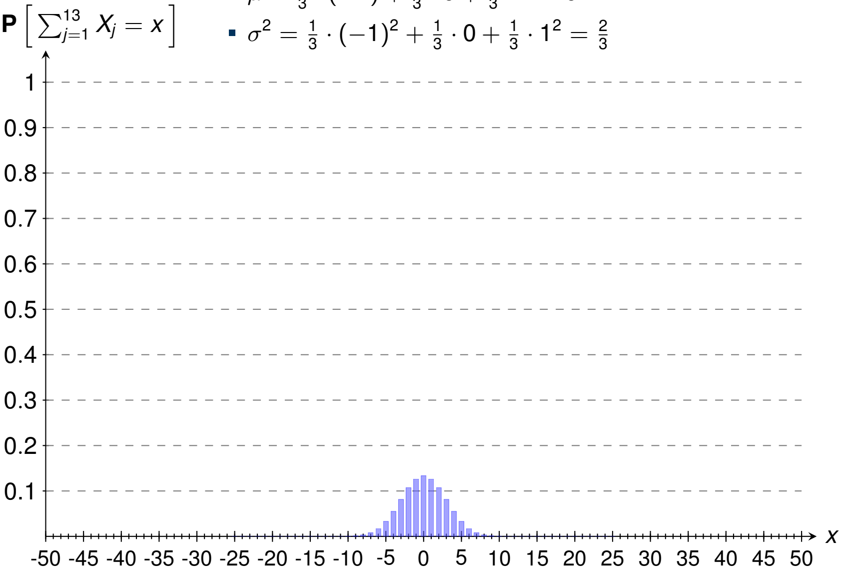

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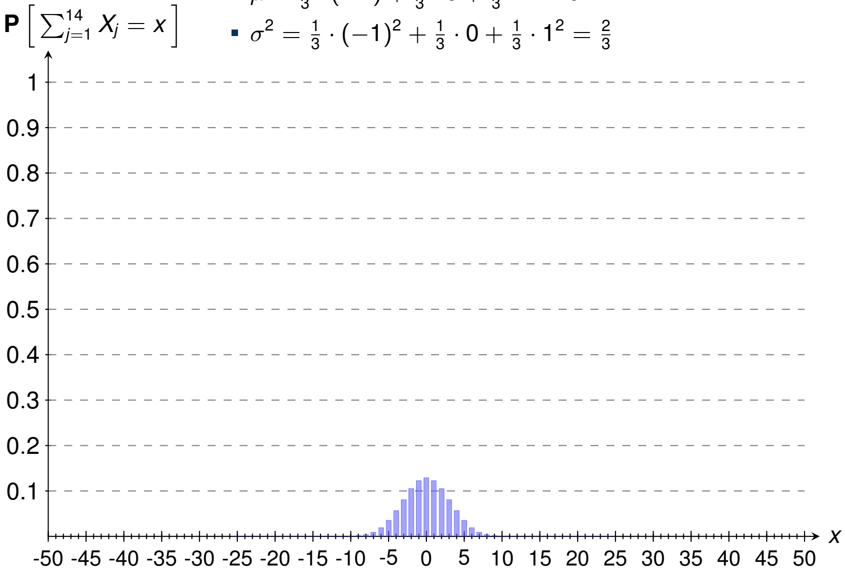

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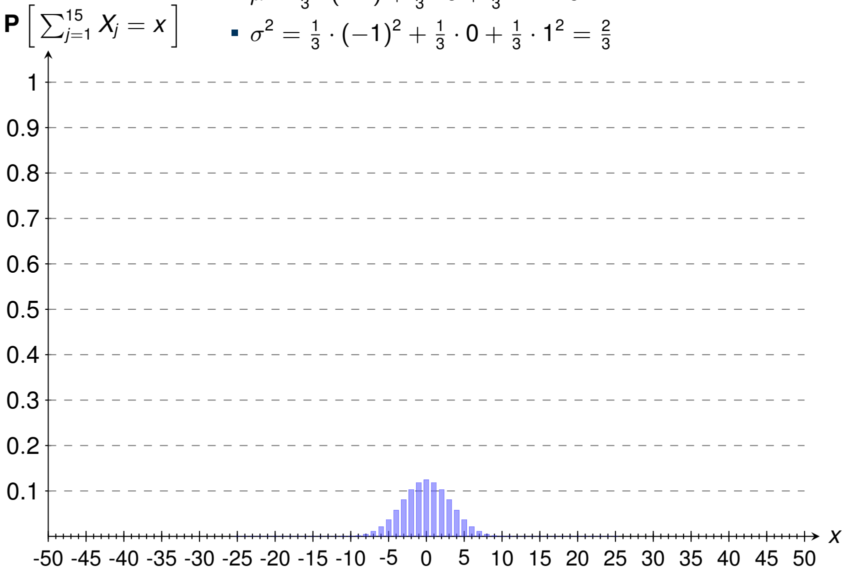

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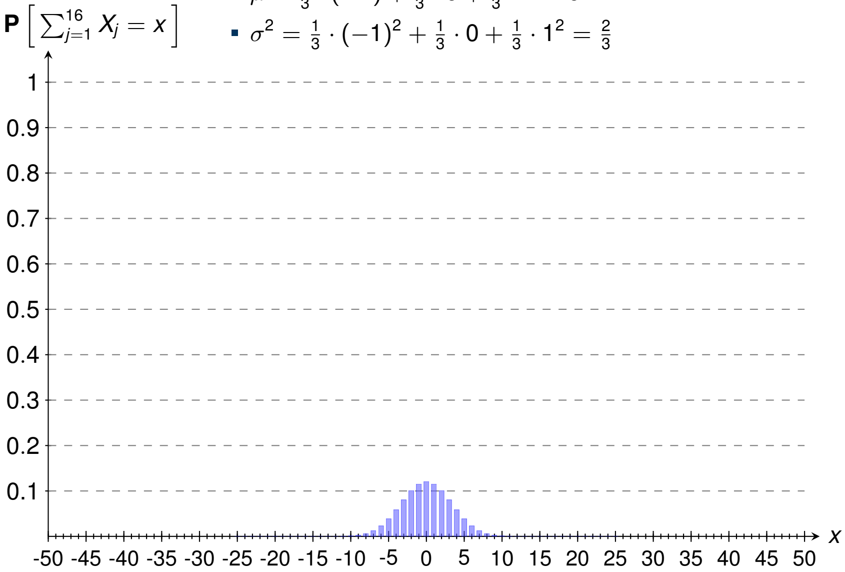

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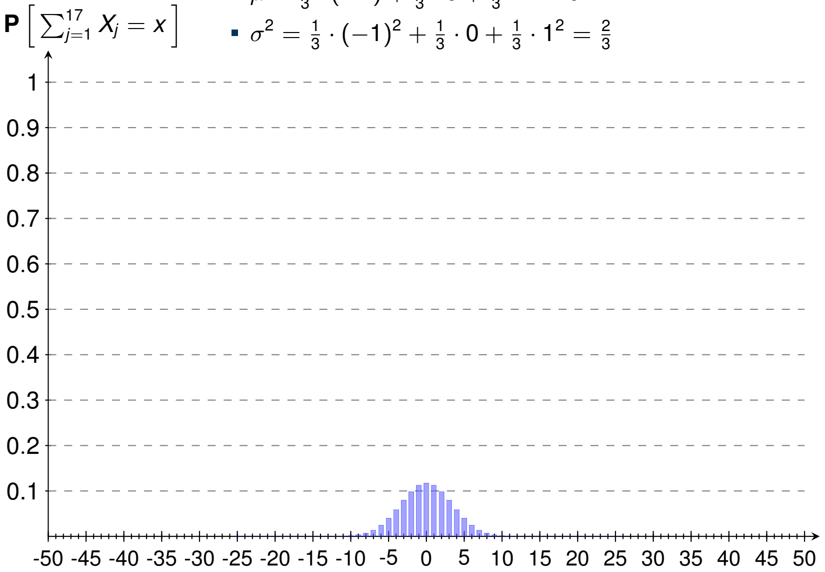

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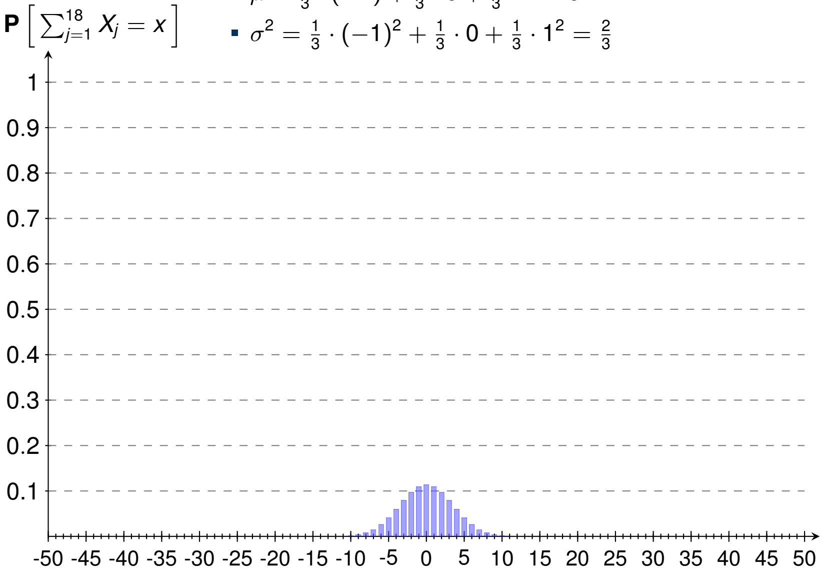

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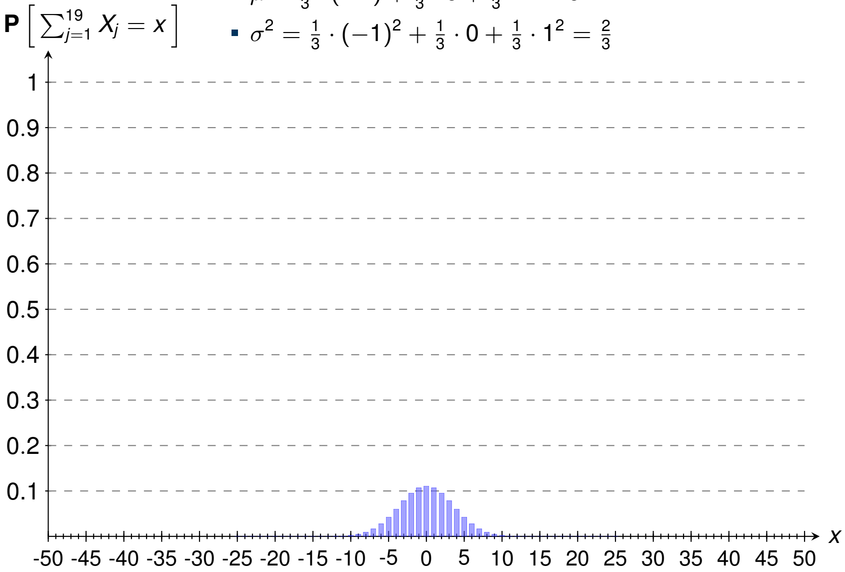

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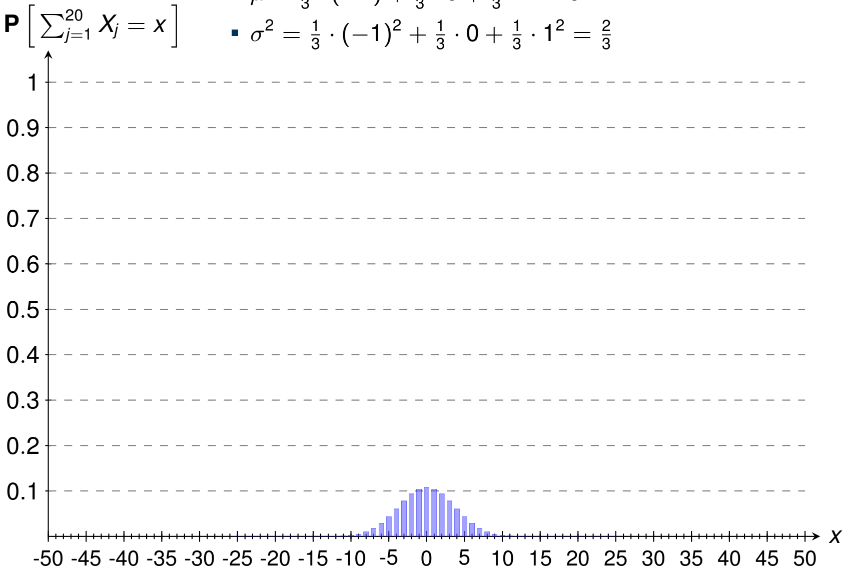

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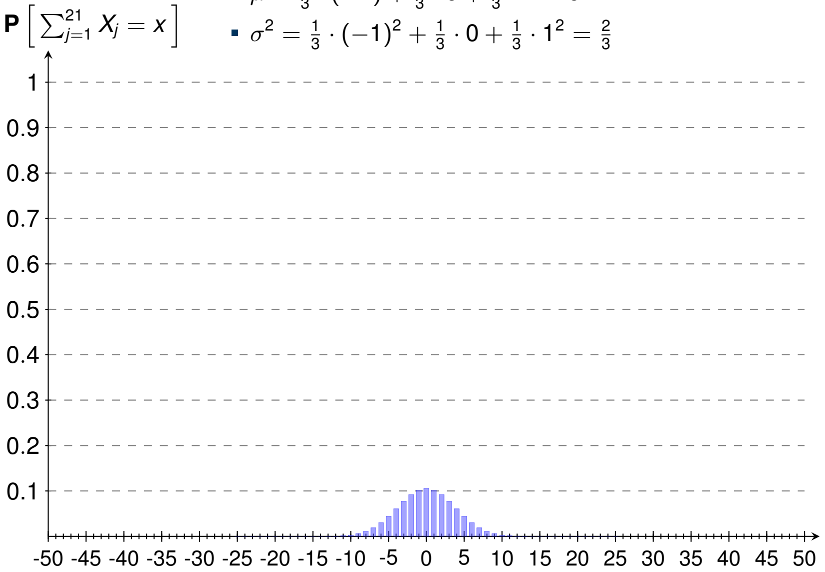

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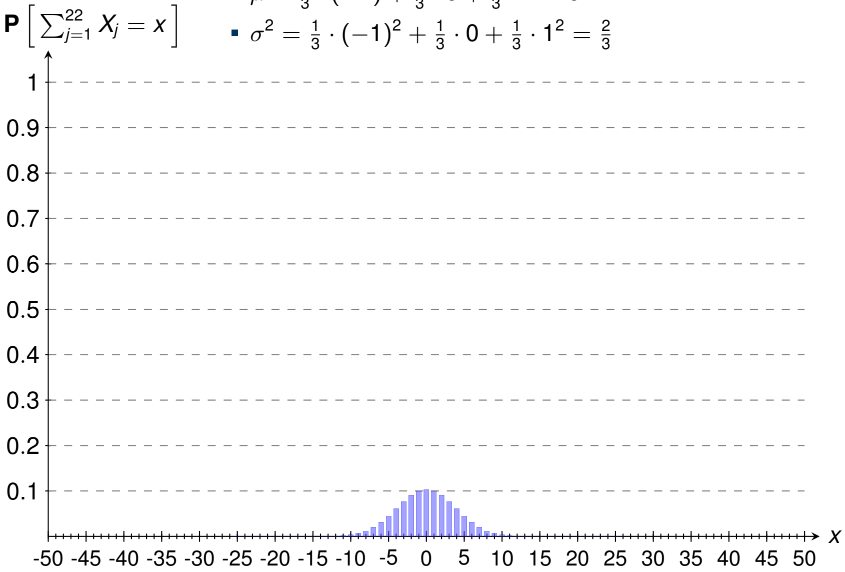

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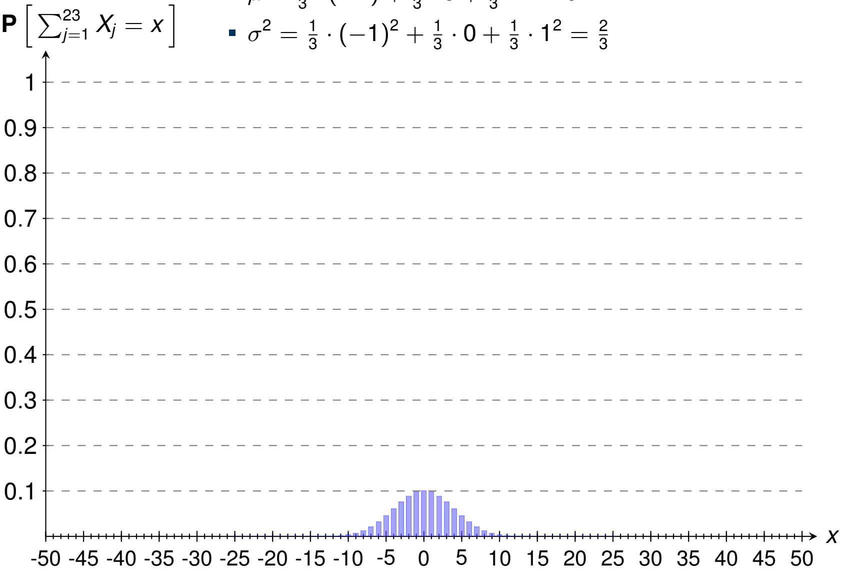

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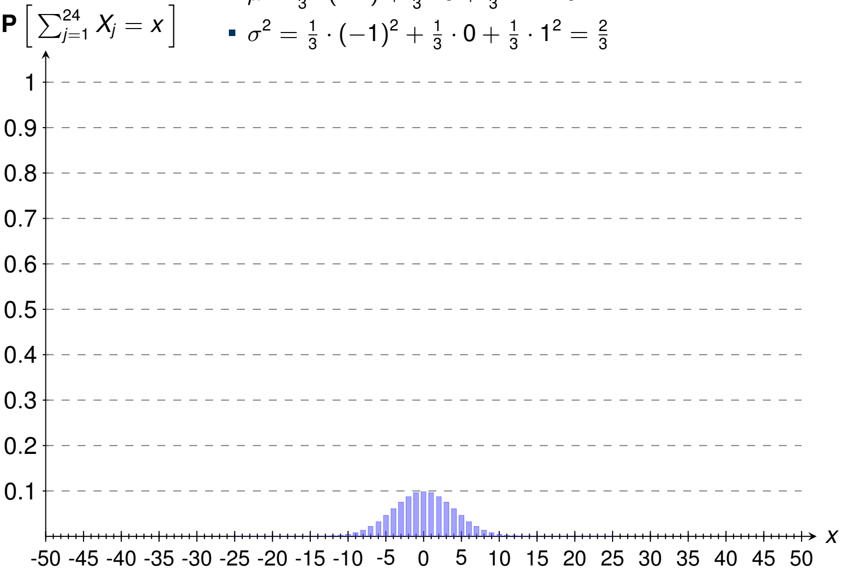

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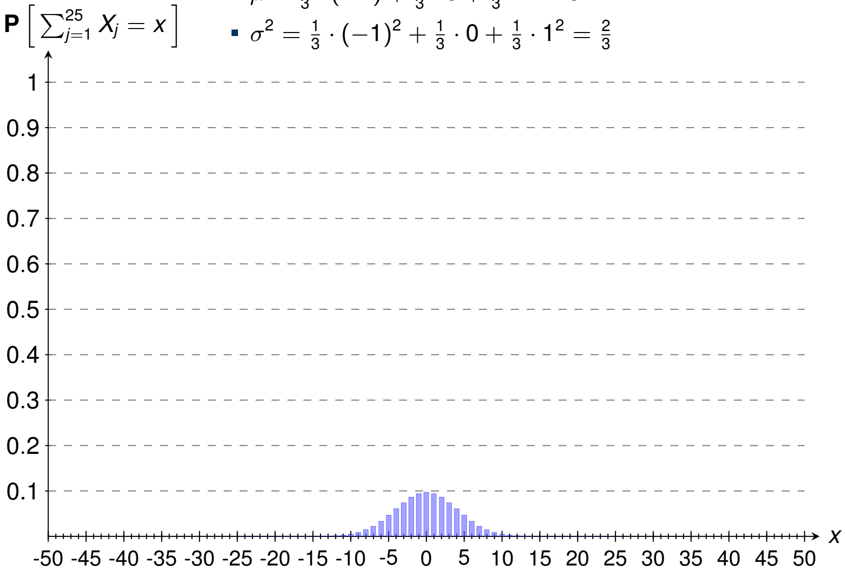

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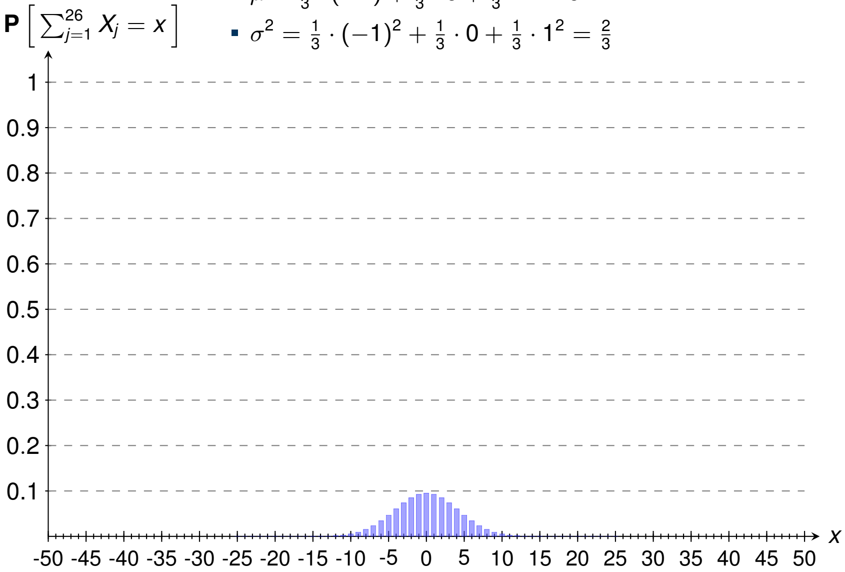

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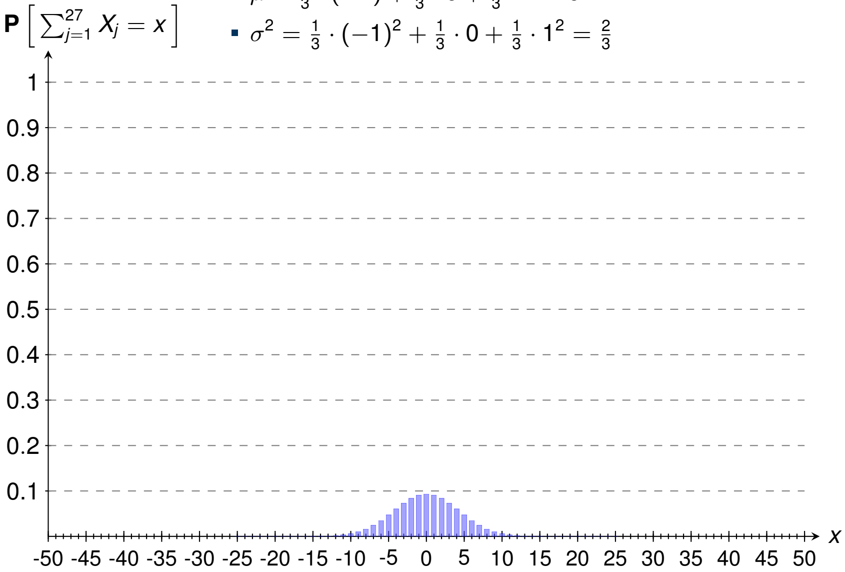

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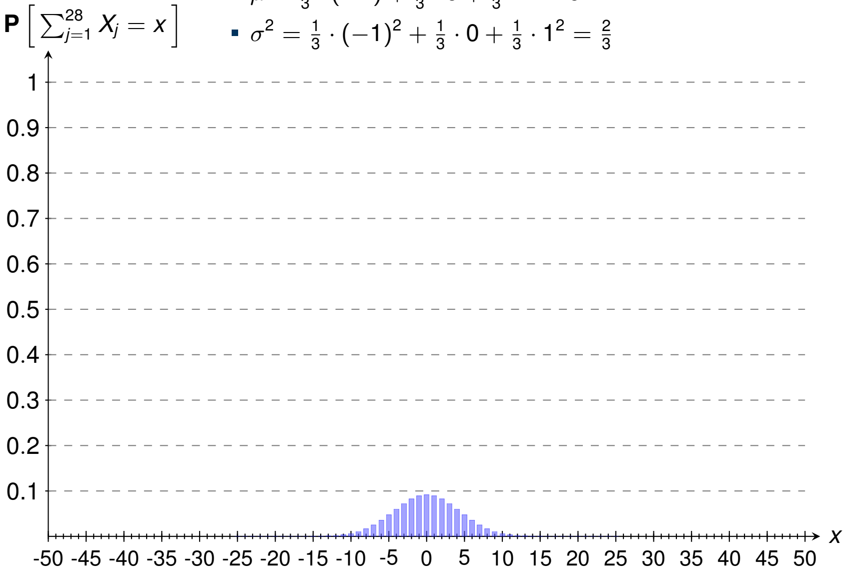

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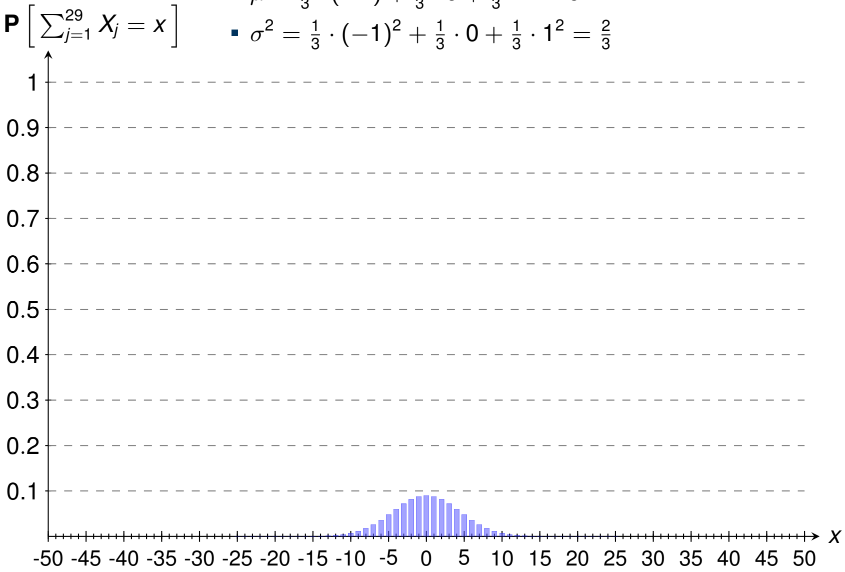

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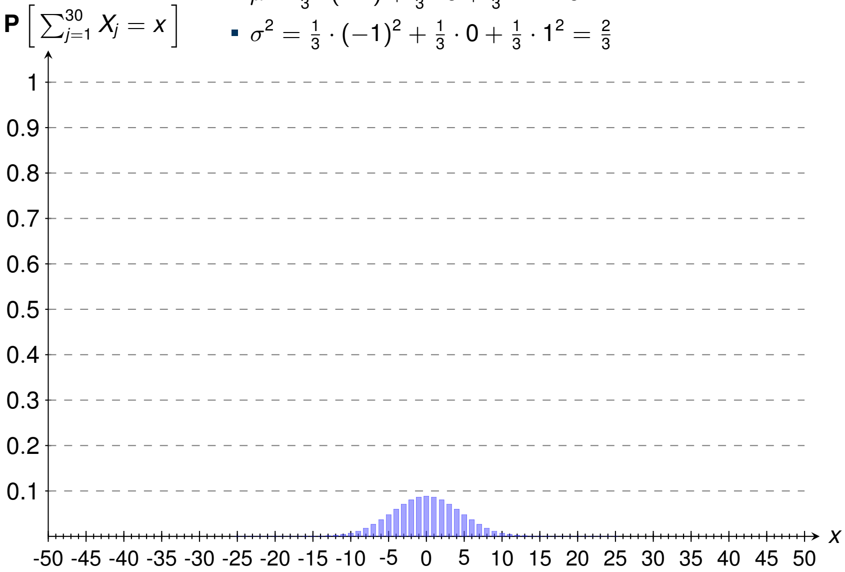

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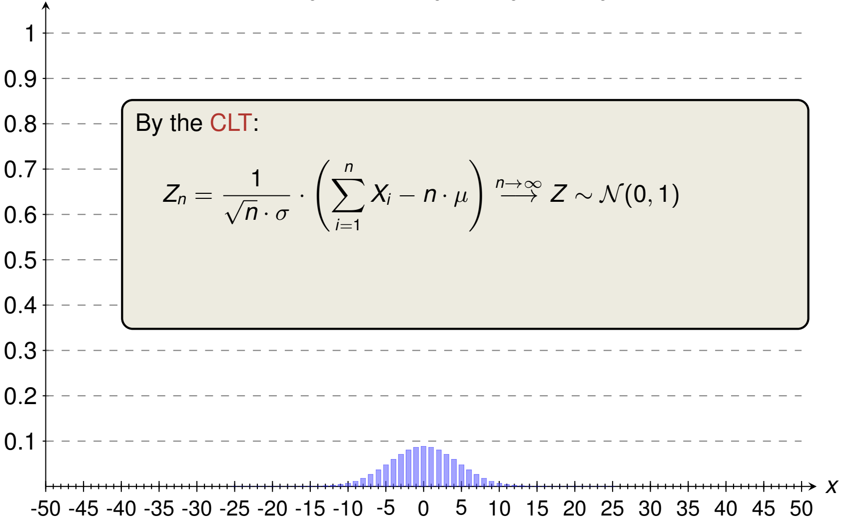
$\mathbf{P}\left[\sum_{j=1}^{30} X_j = x\right]$



By the CLT:

$$Z_n = \frac{1}{\sqrt{n} \cdot \sigma} \cdot \left(\sum_{i=1}^{n} X_i - n \cdot \mu \right) \xrightarrow{n \to \infty} Z \sim \mathcal{N}(0, 1)$$

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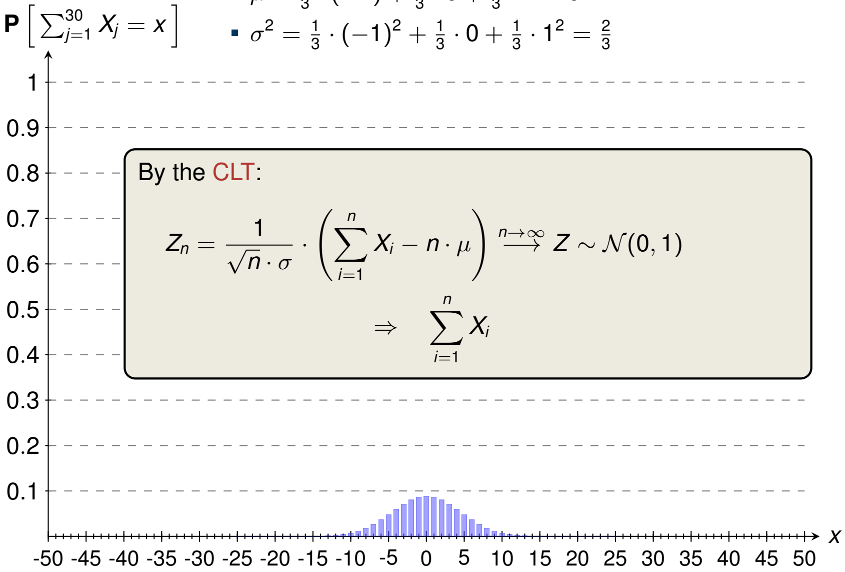

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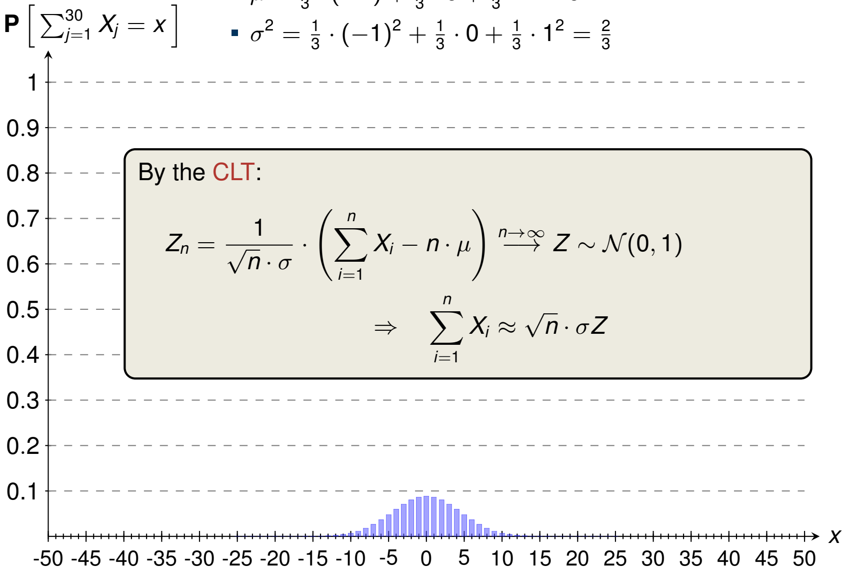

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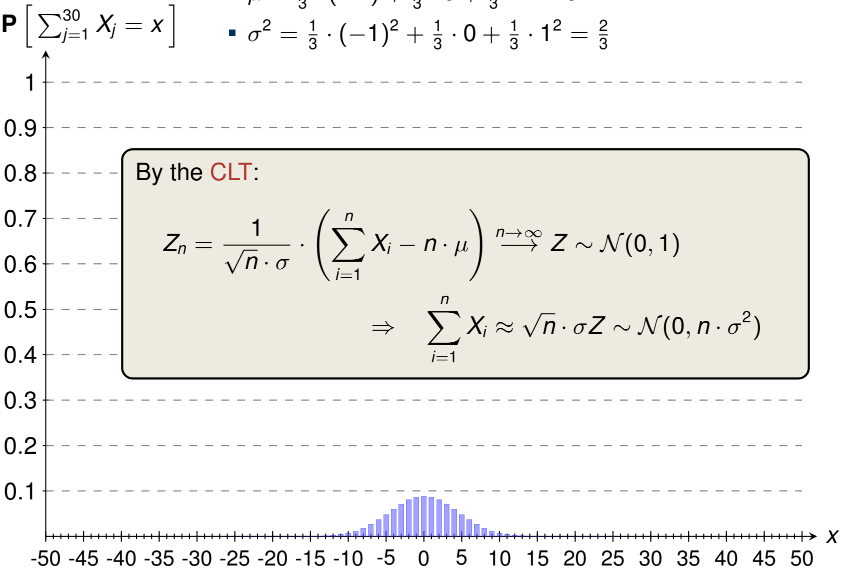

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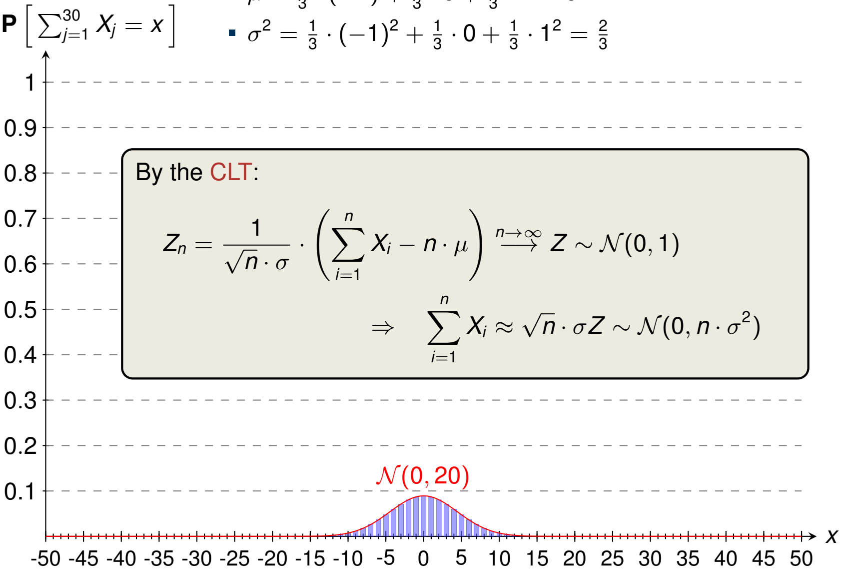

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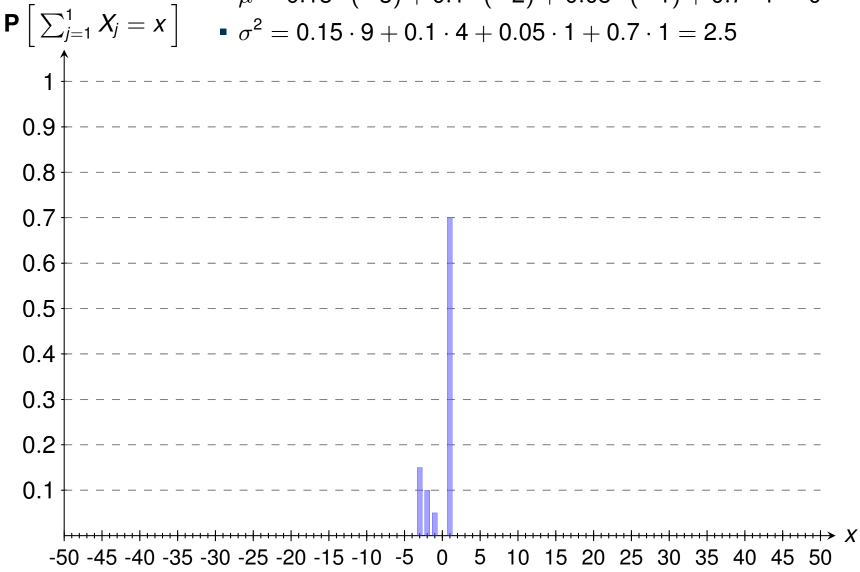

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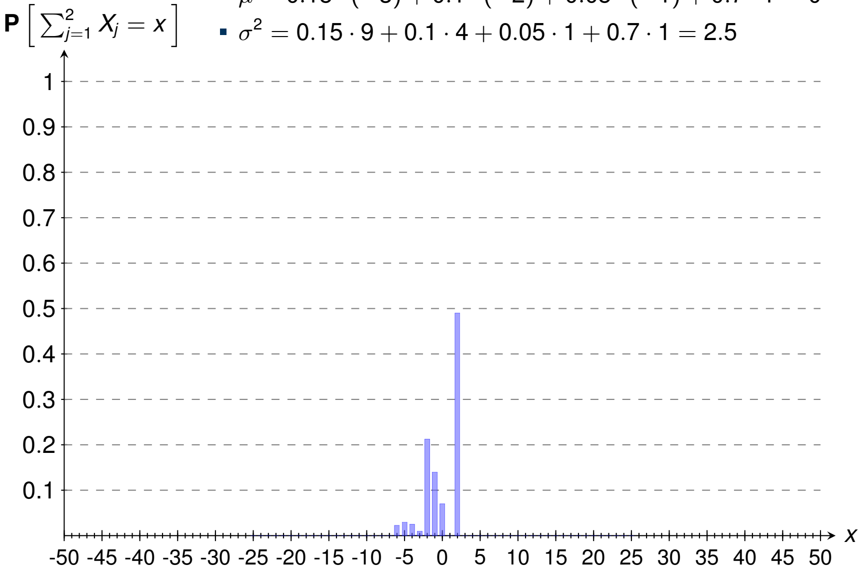

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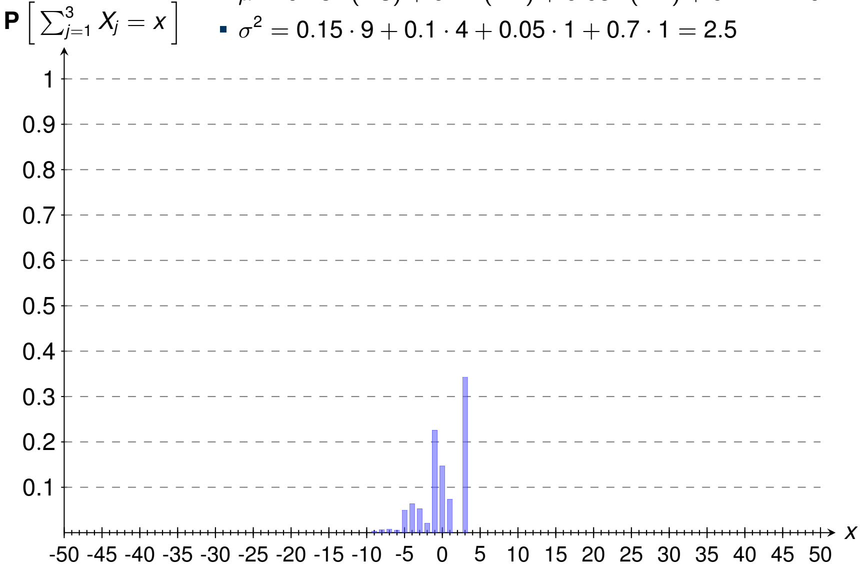

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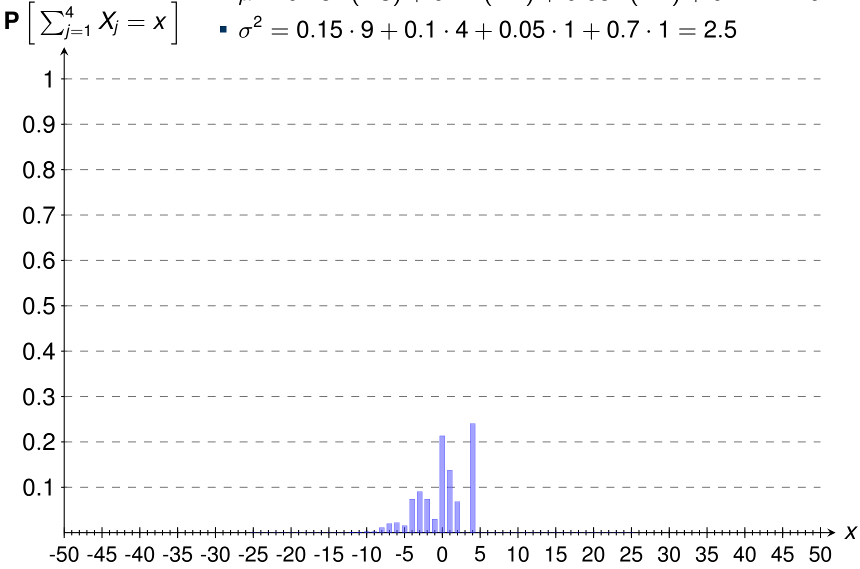

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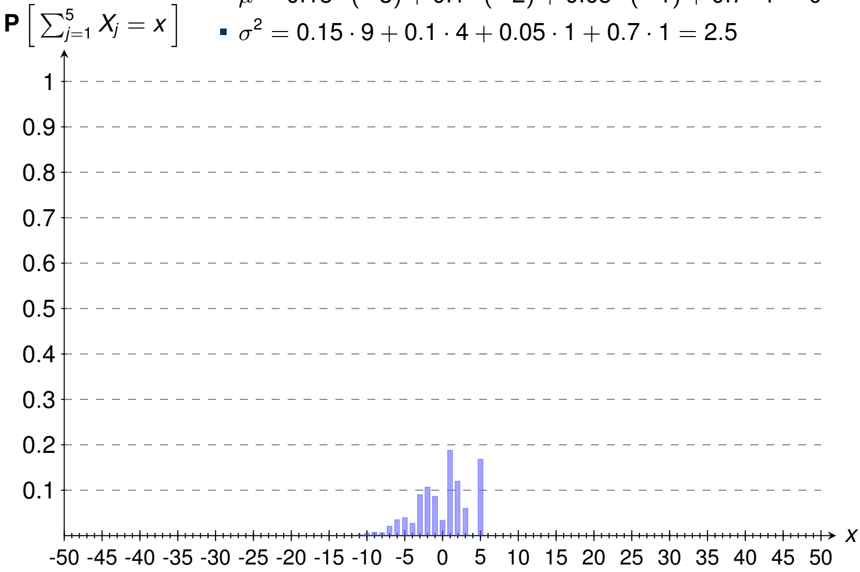

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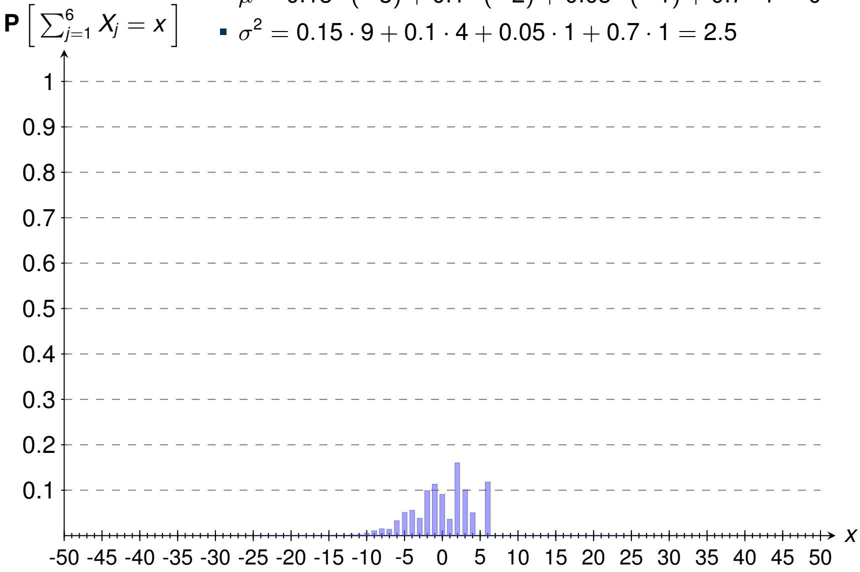

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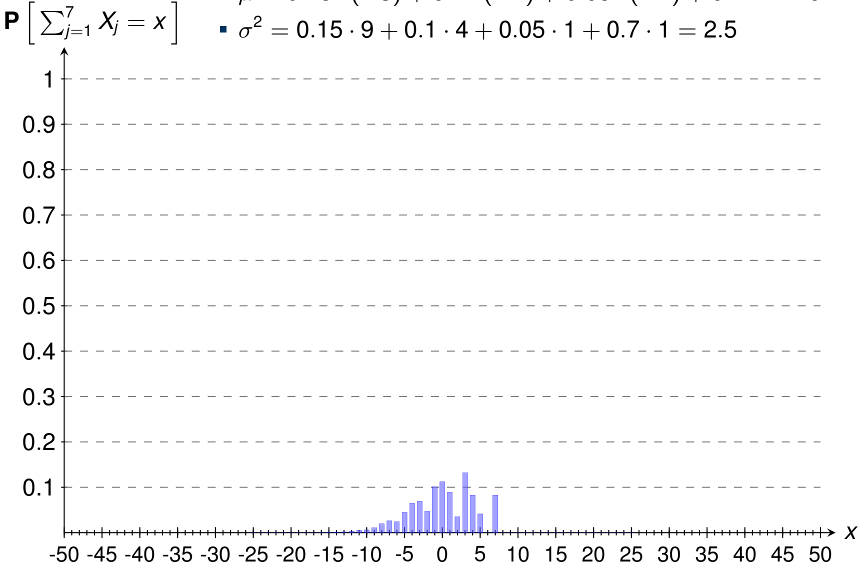

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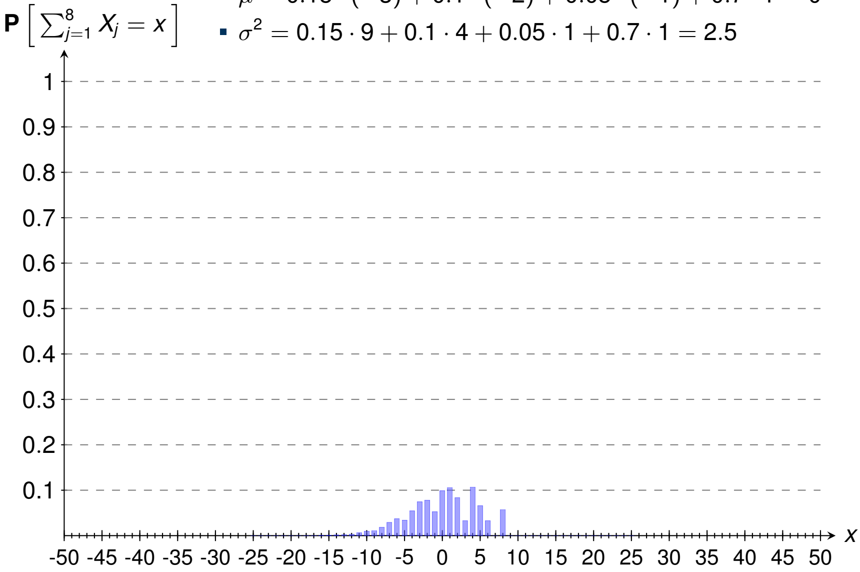

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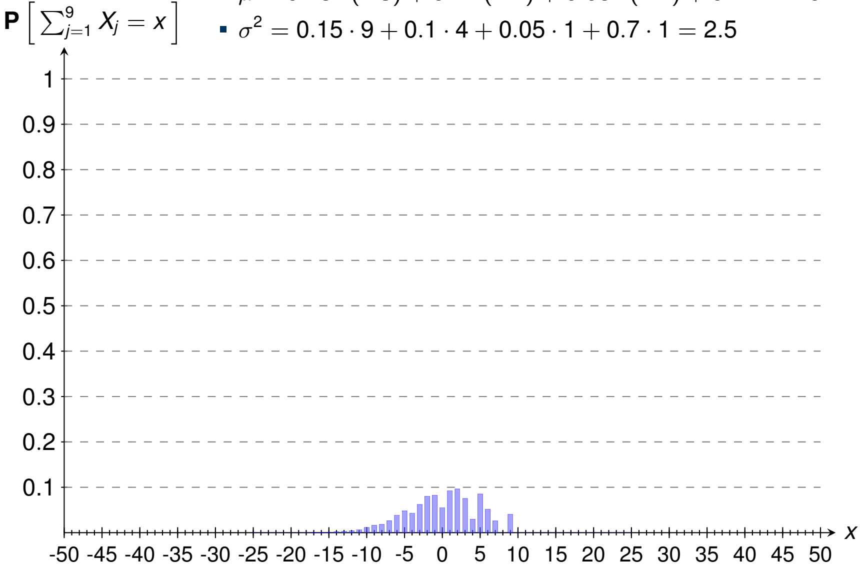

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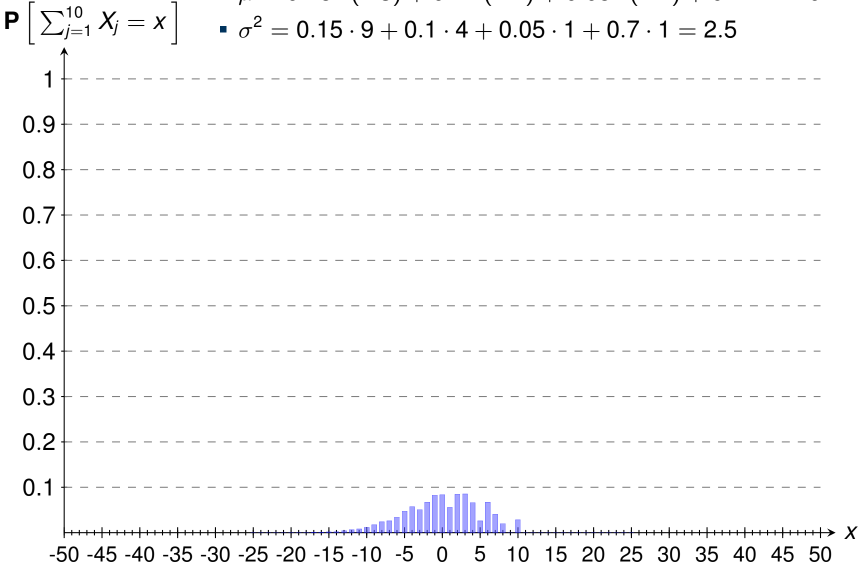

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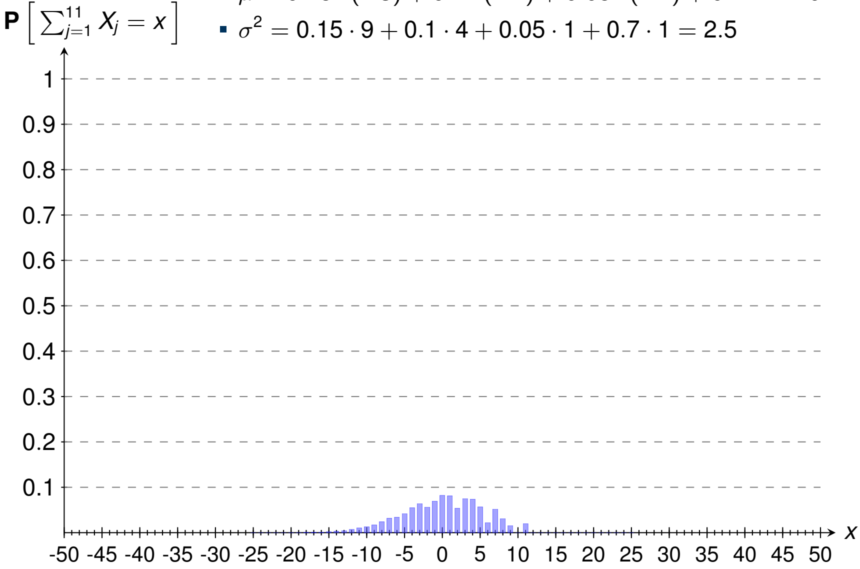

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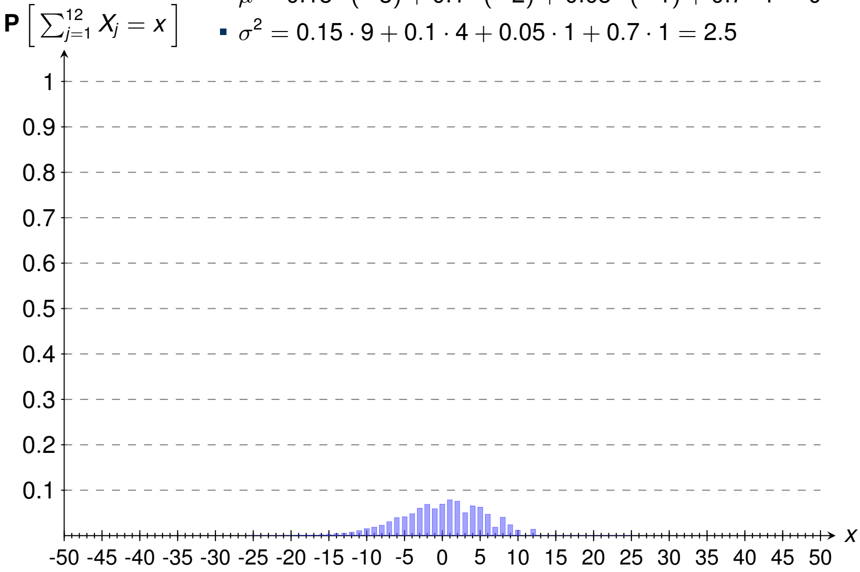

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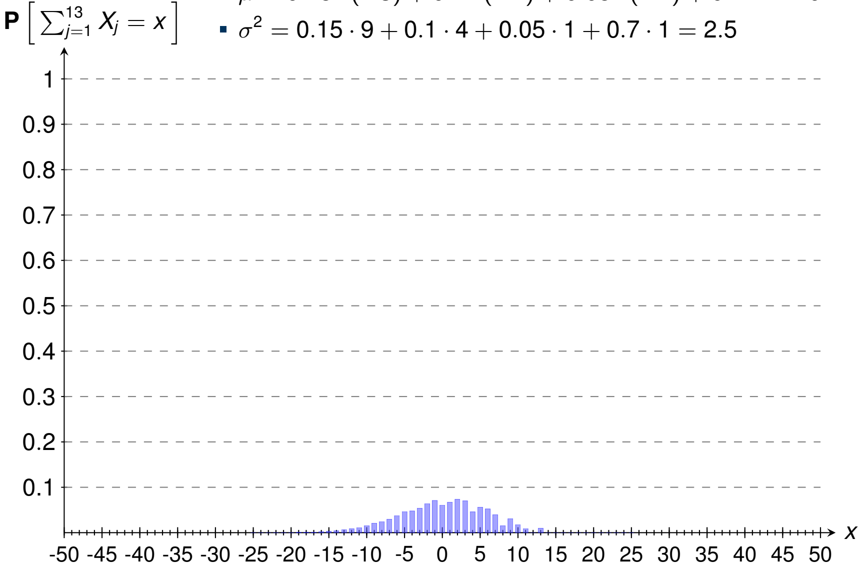

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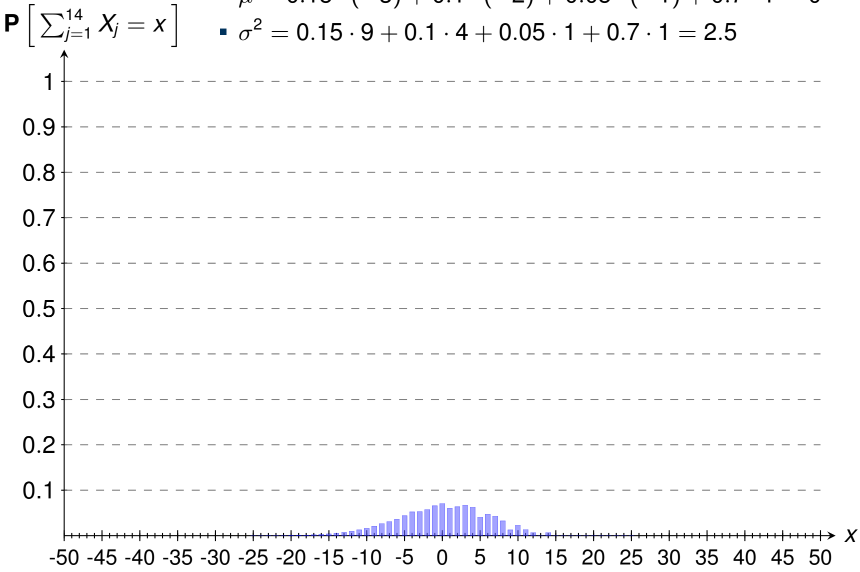

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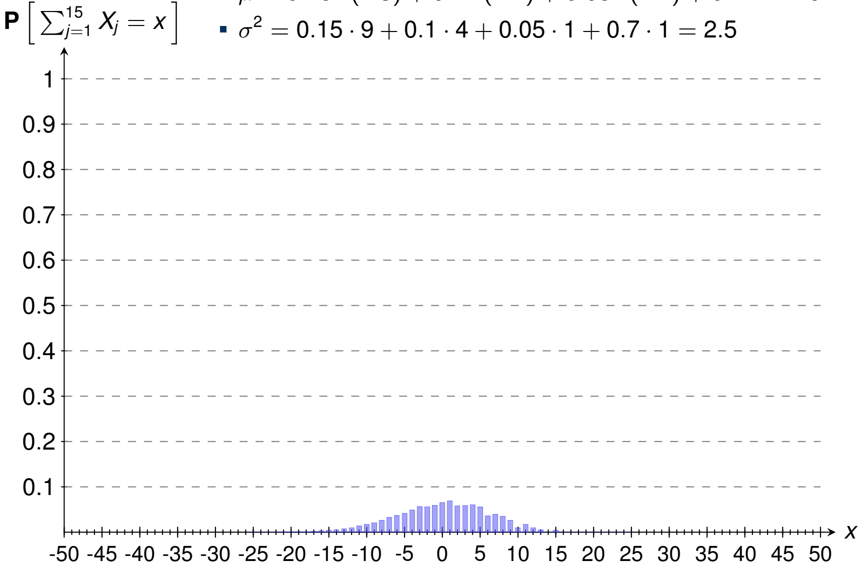

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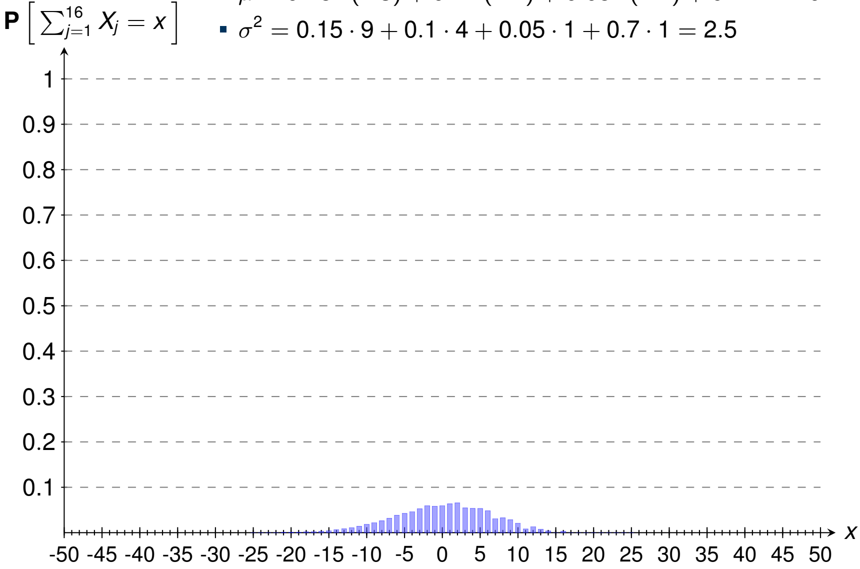

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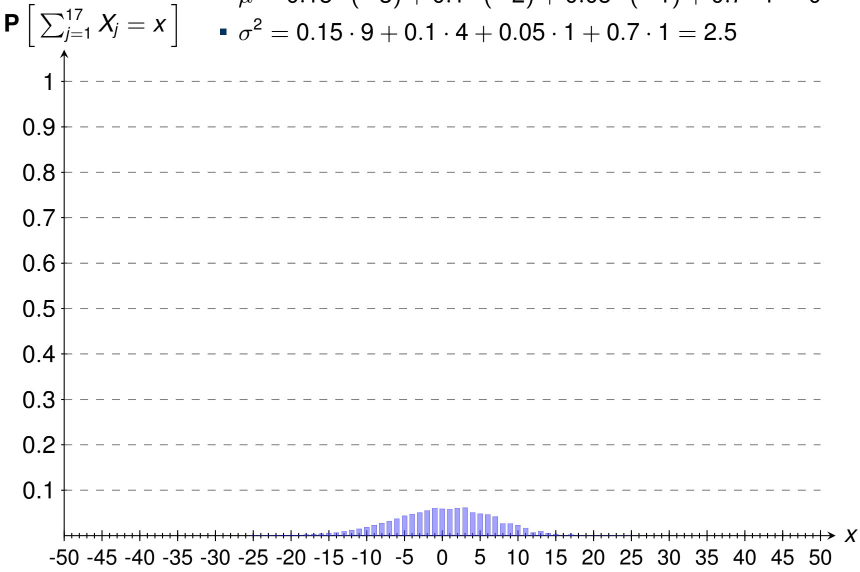

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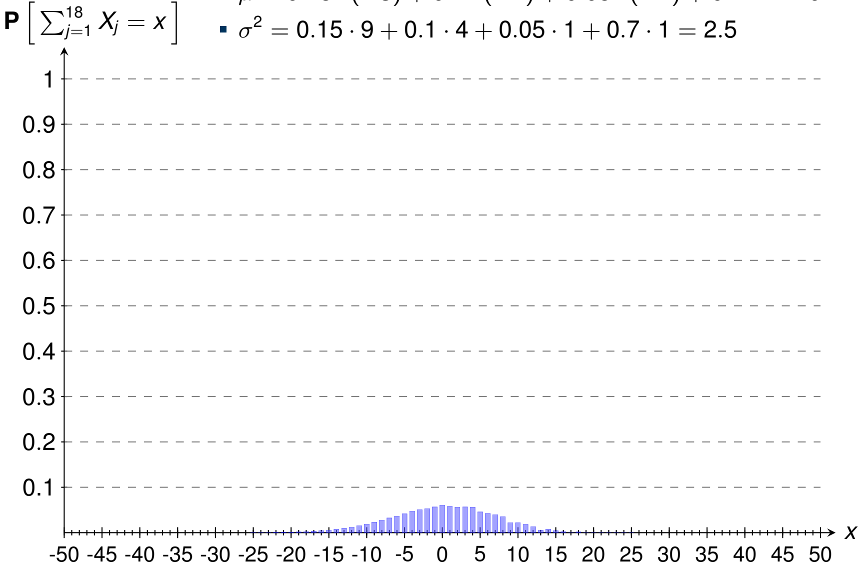

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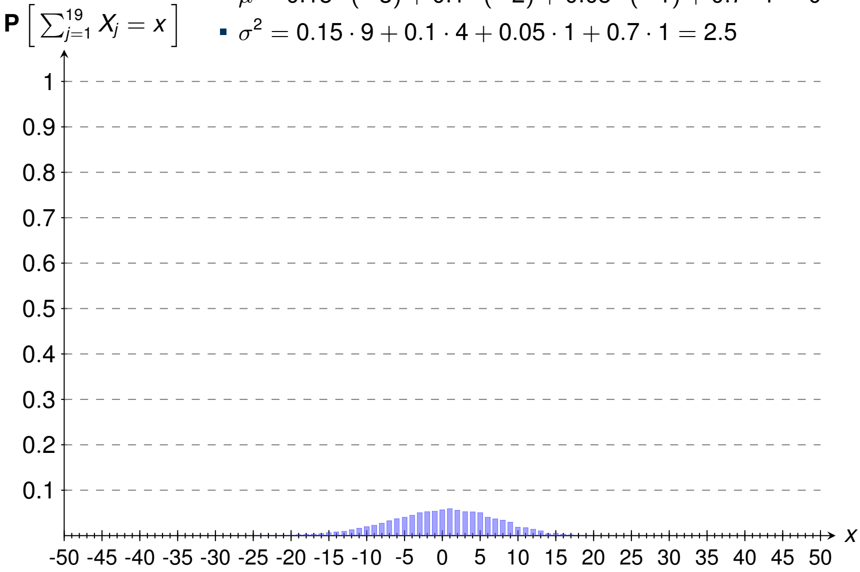

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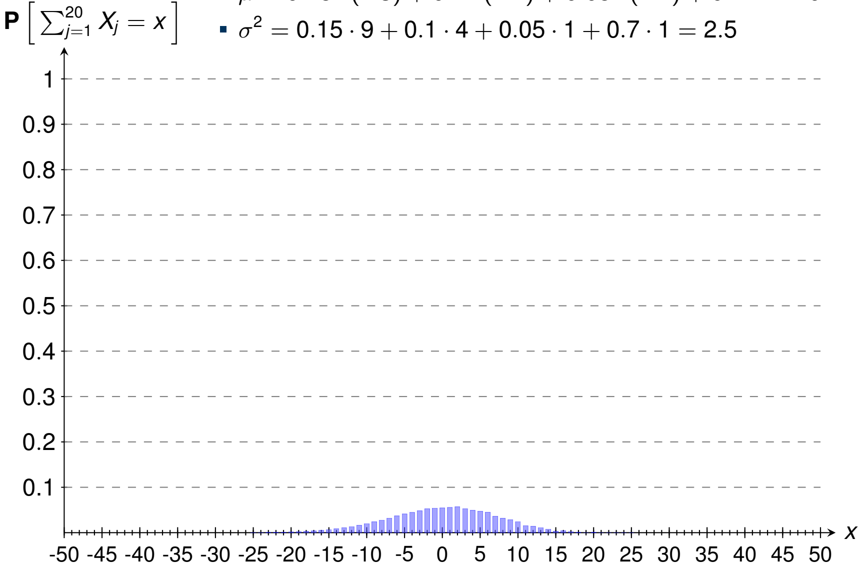

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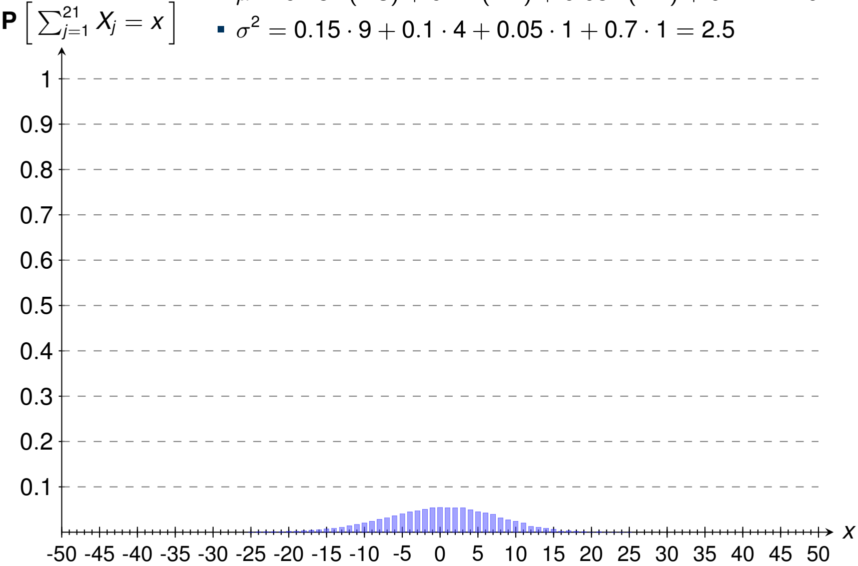

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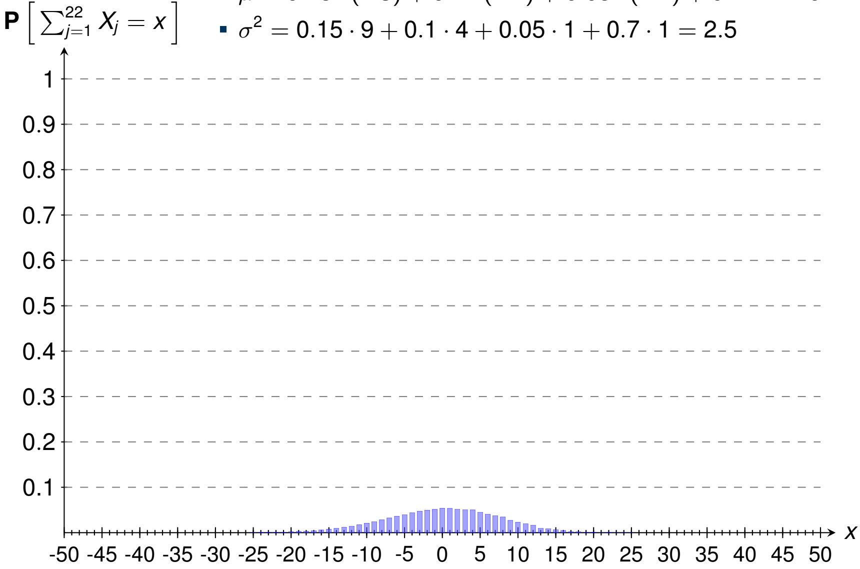

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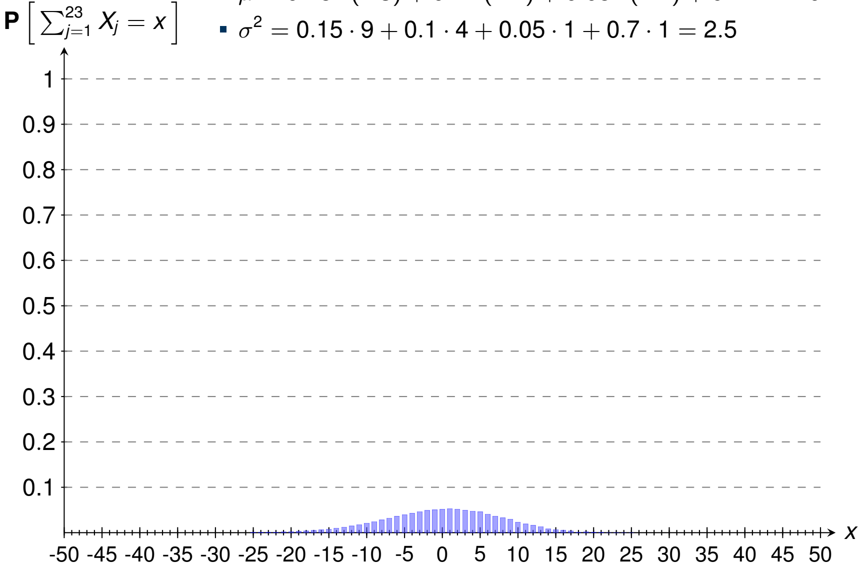

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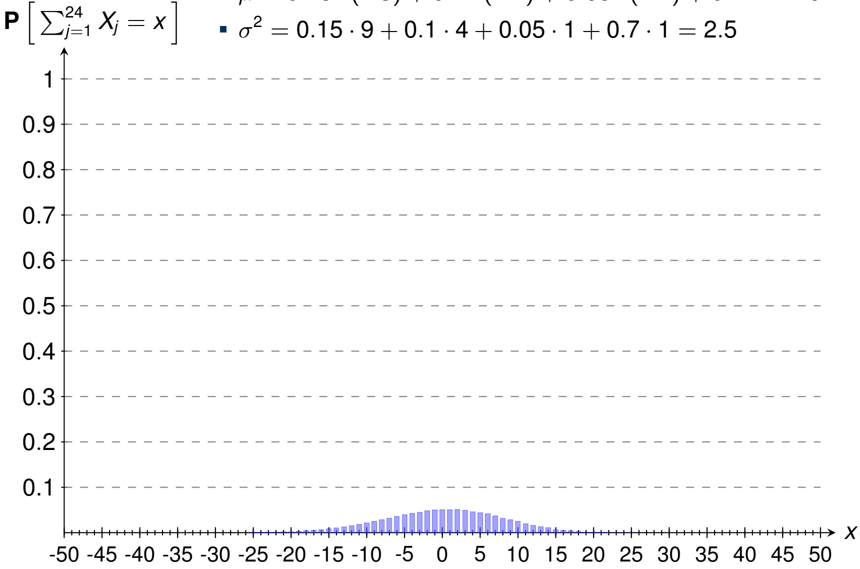

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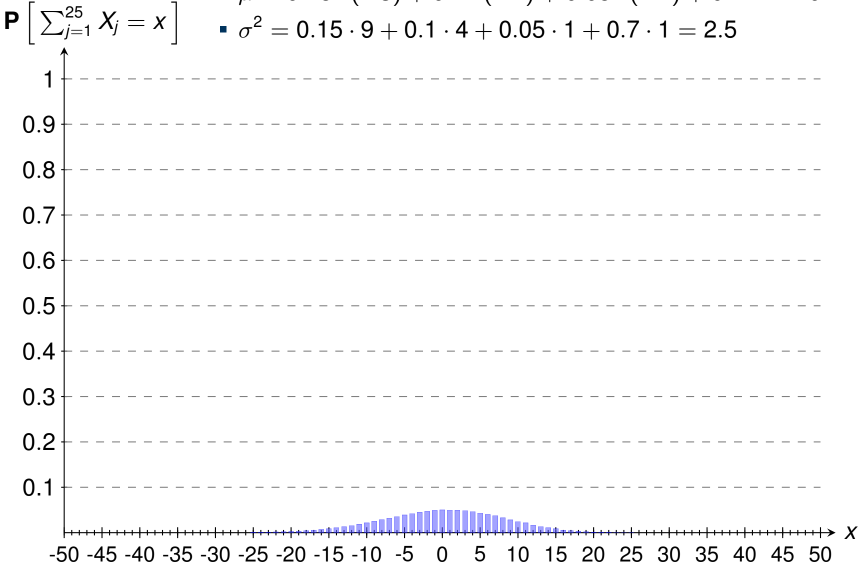

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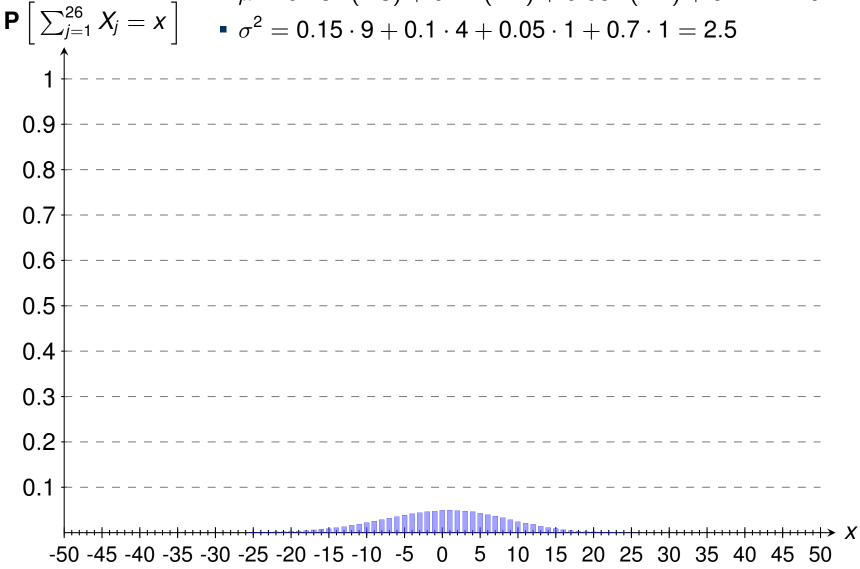

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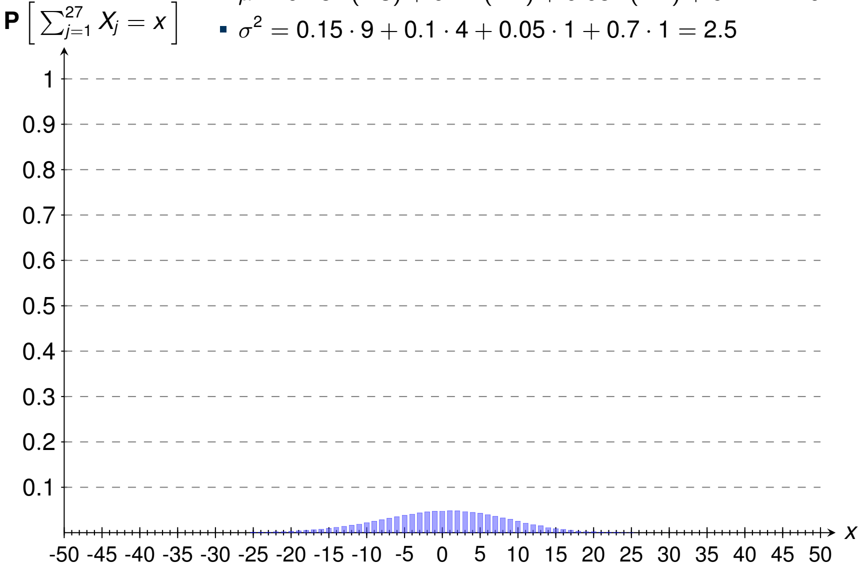

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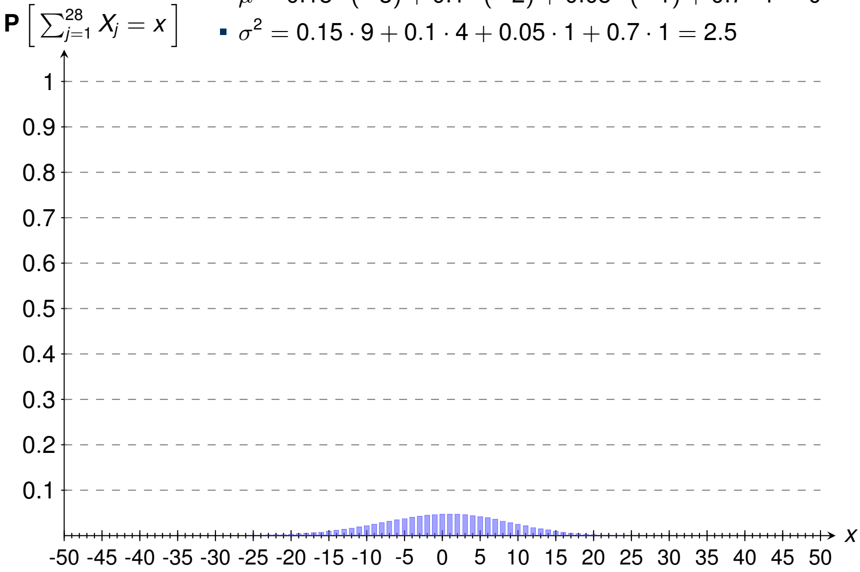

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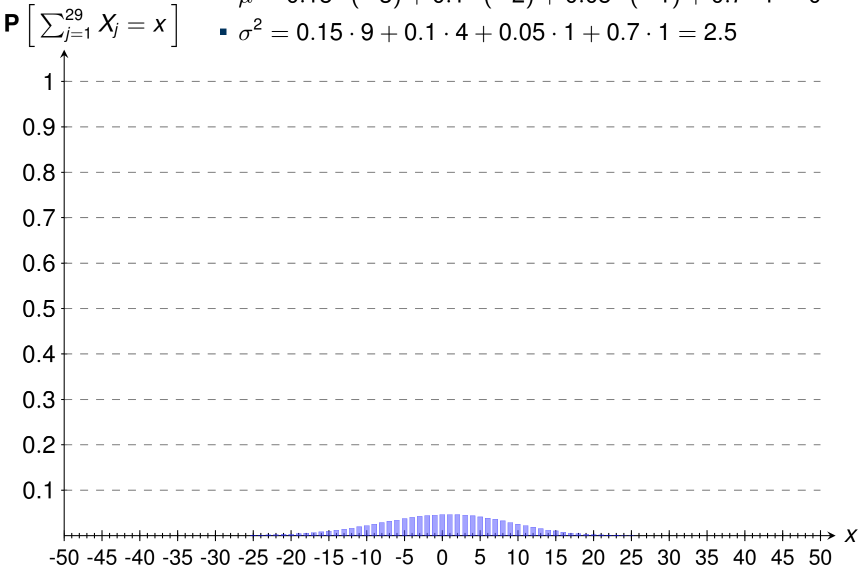

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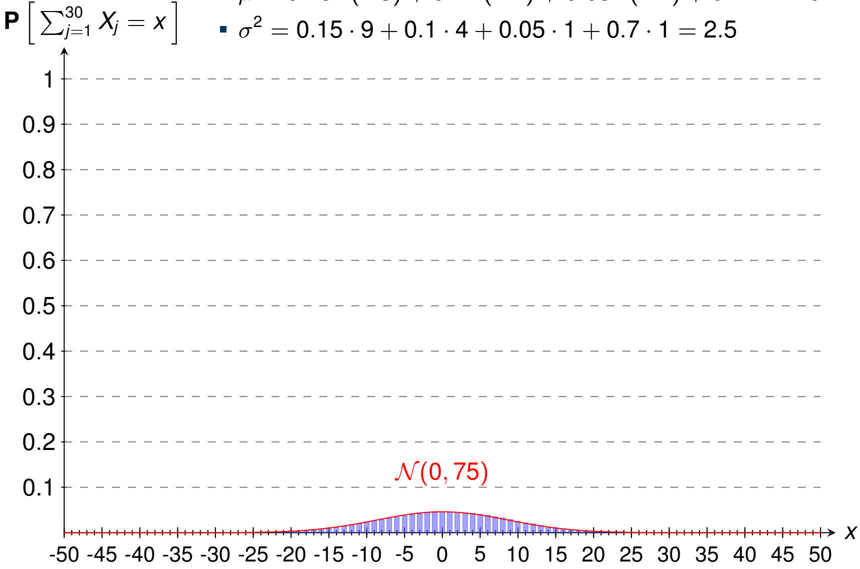

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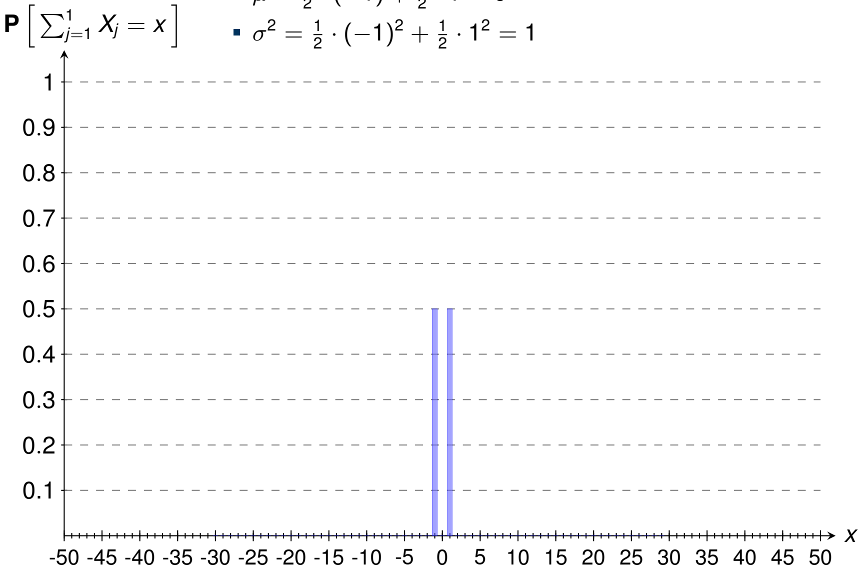

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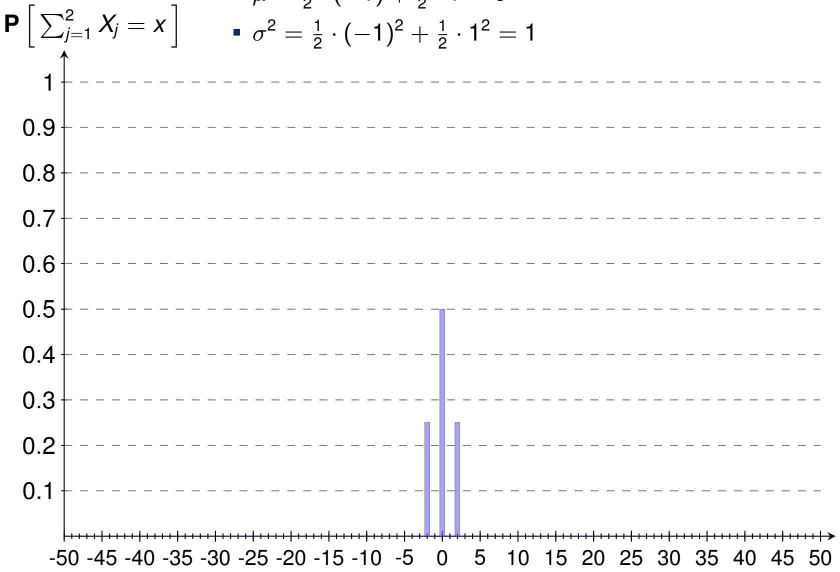

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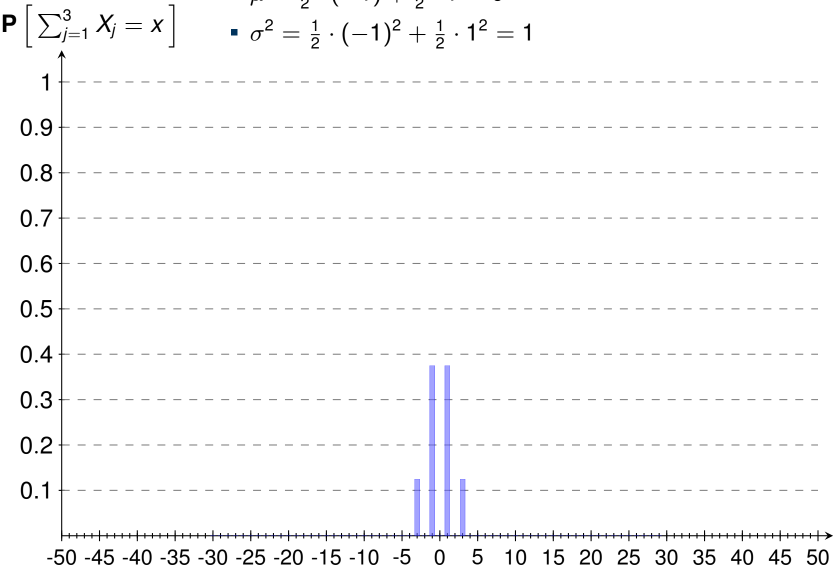

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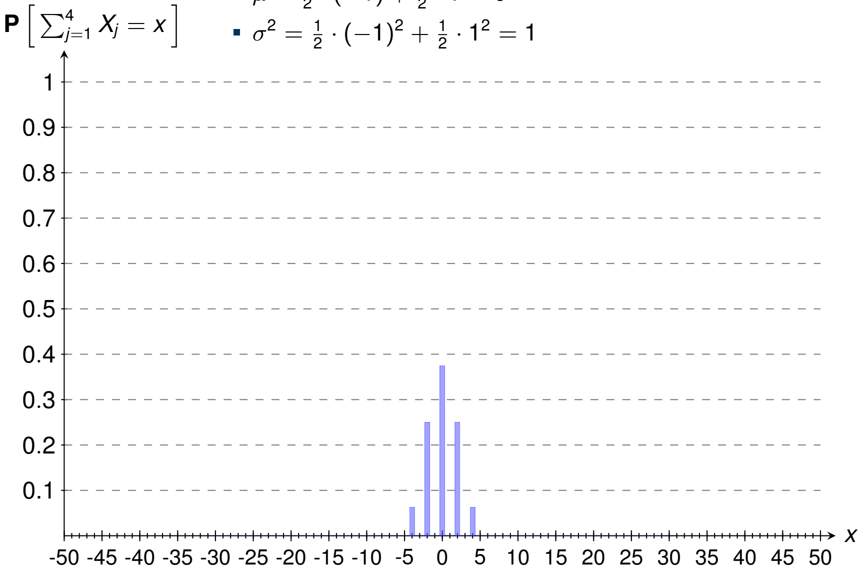

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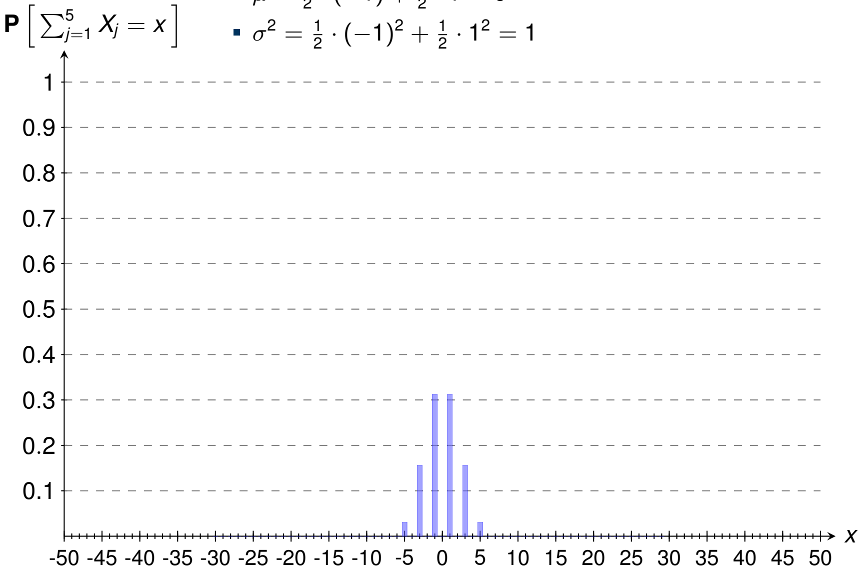

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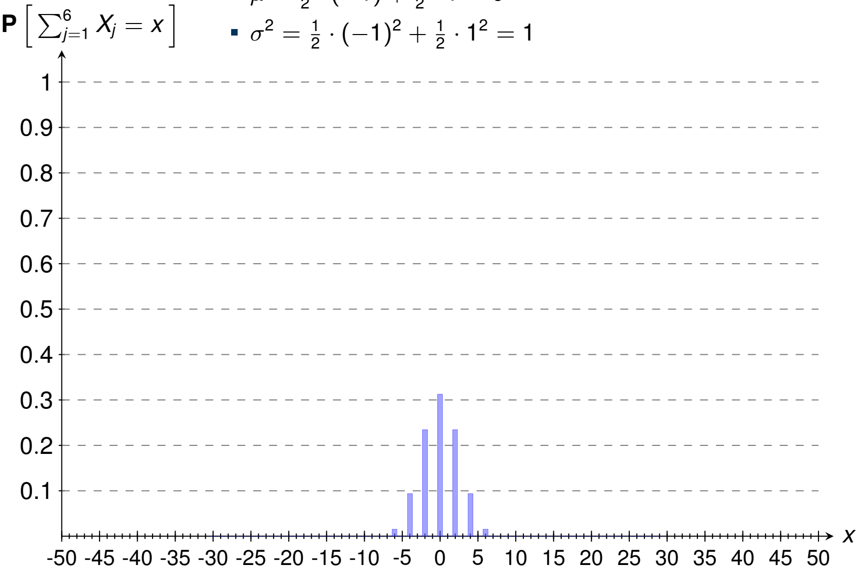

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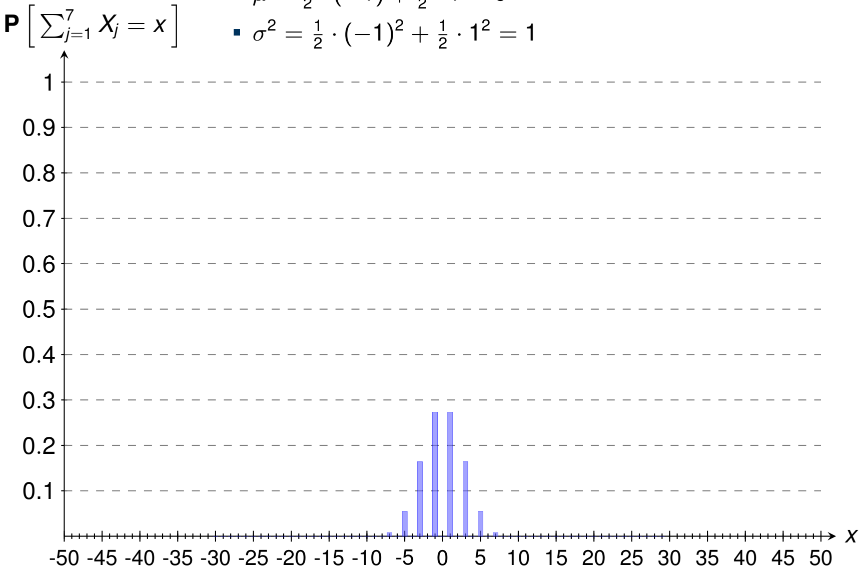

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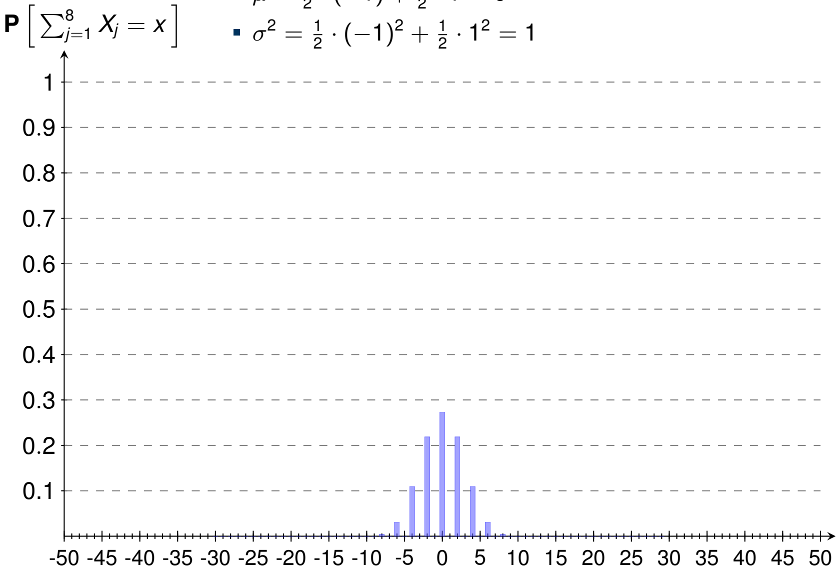

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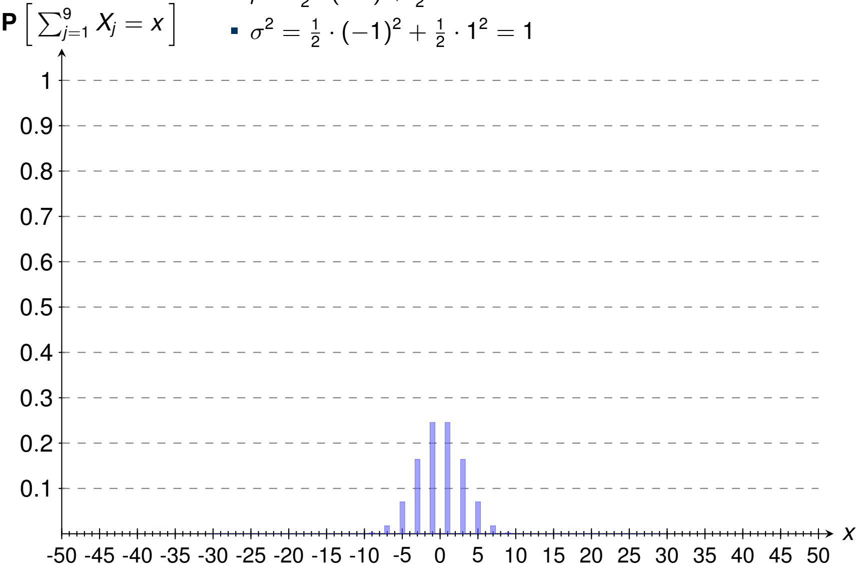

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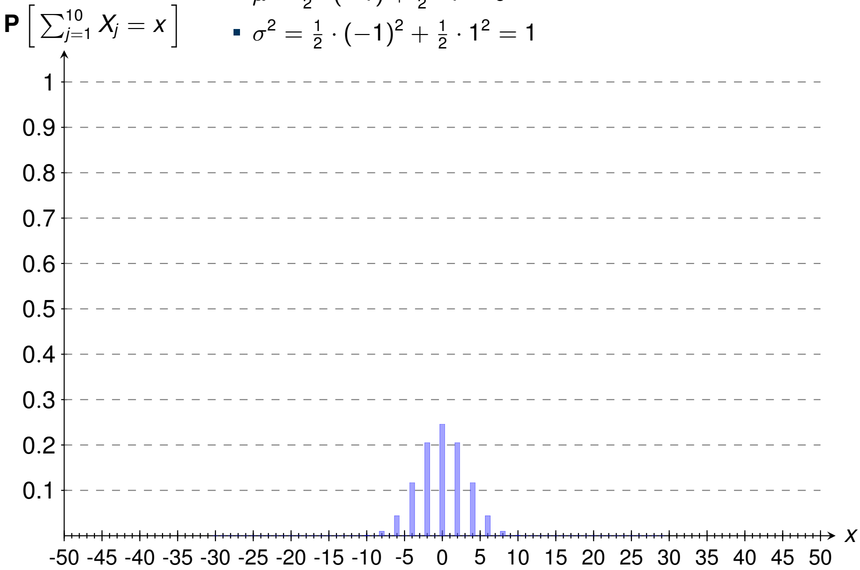

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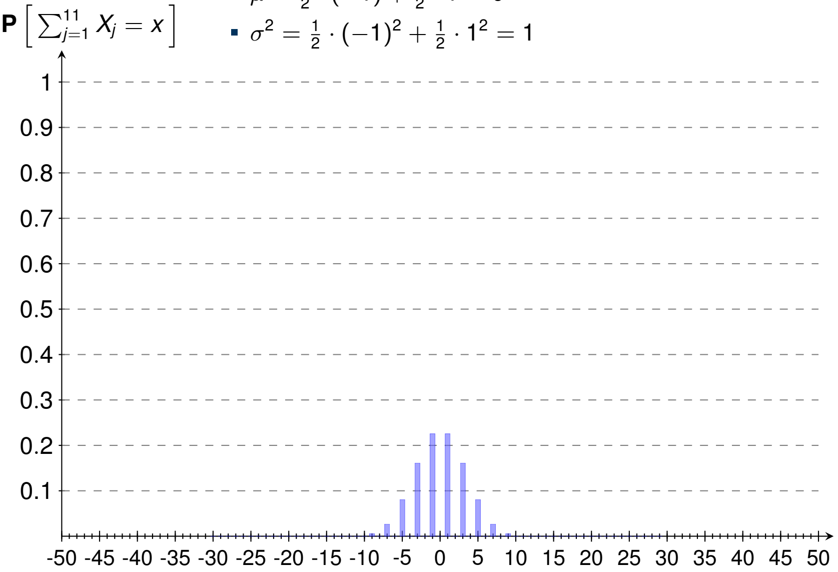

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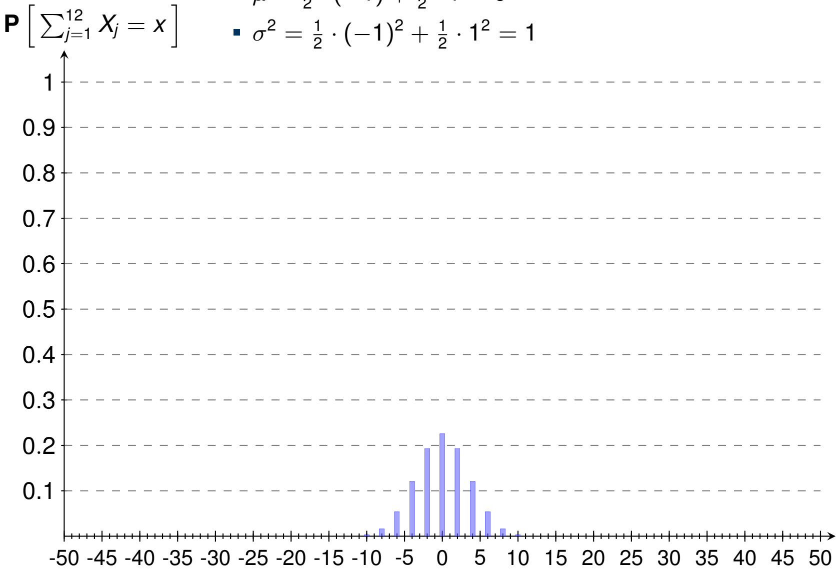

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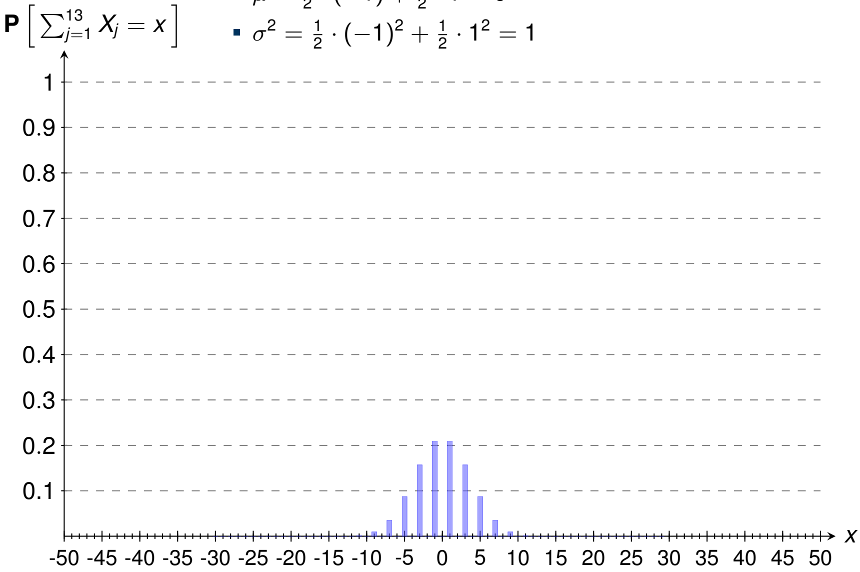

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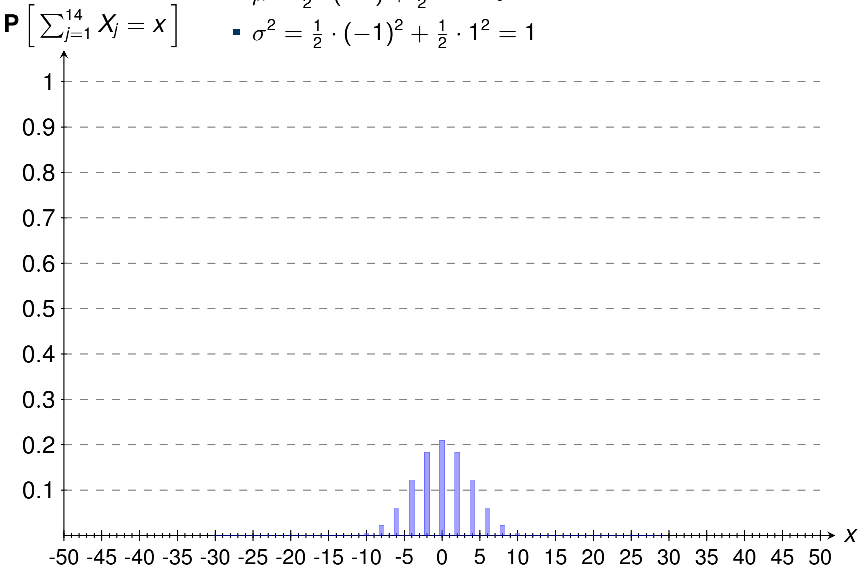

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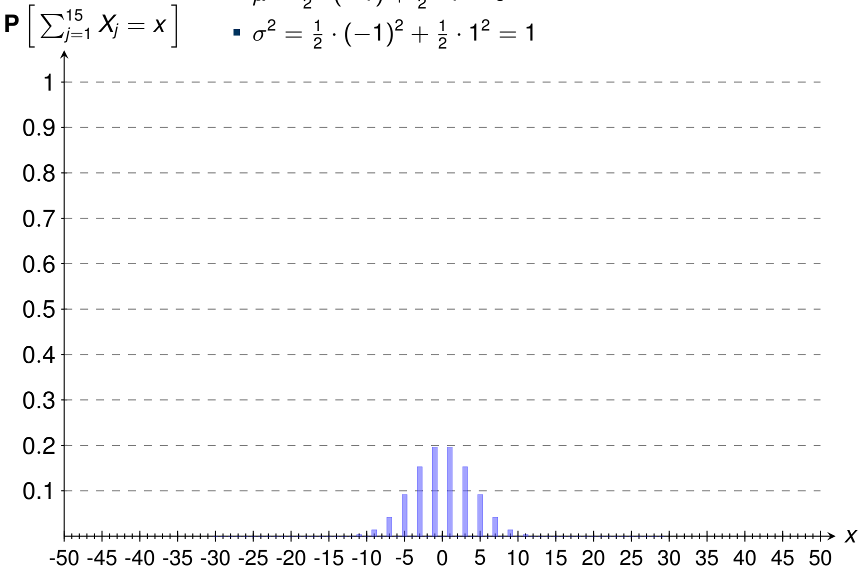

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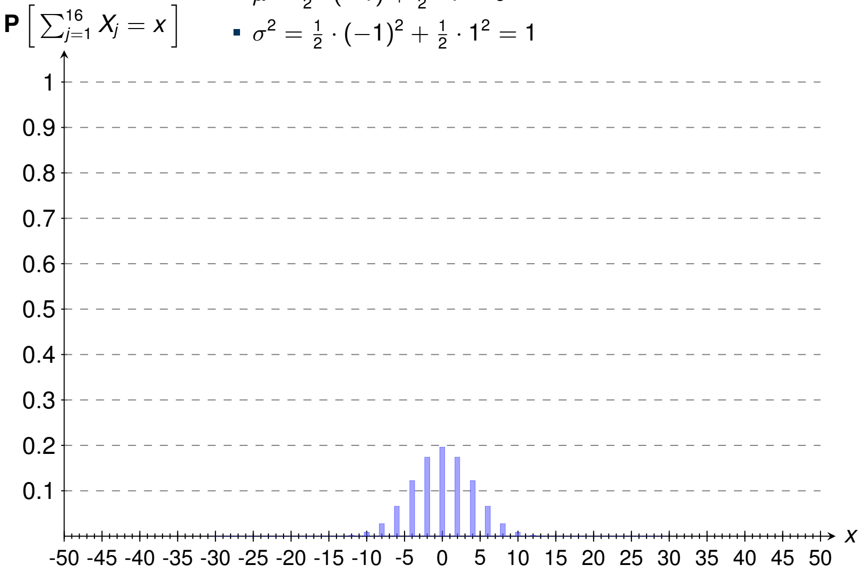

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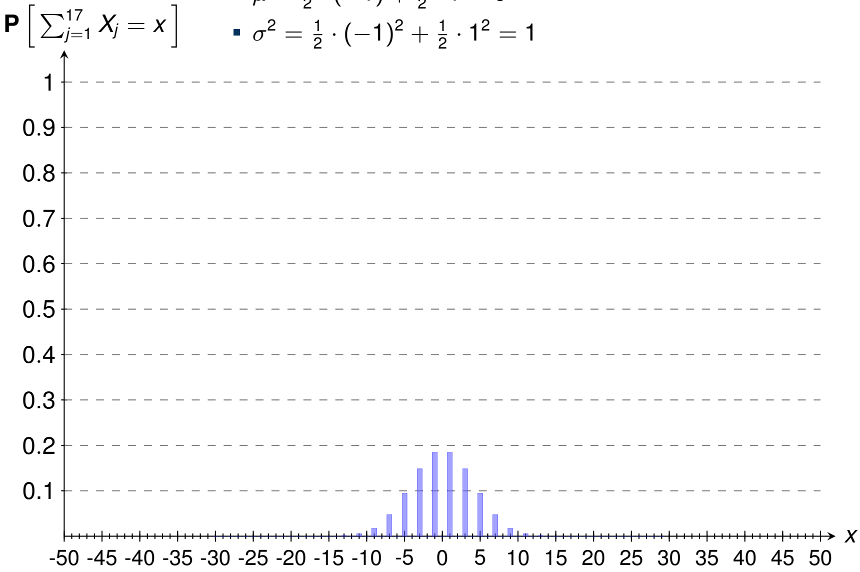

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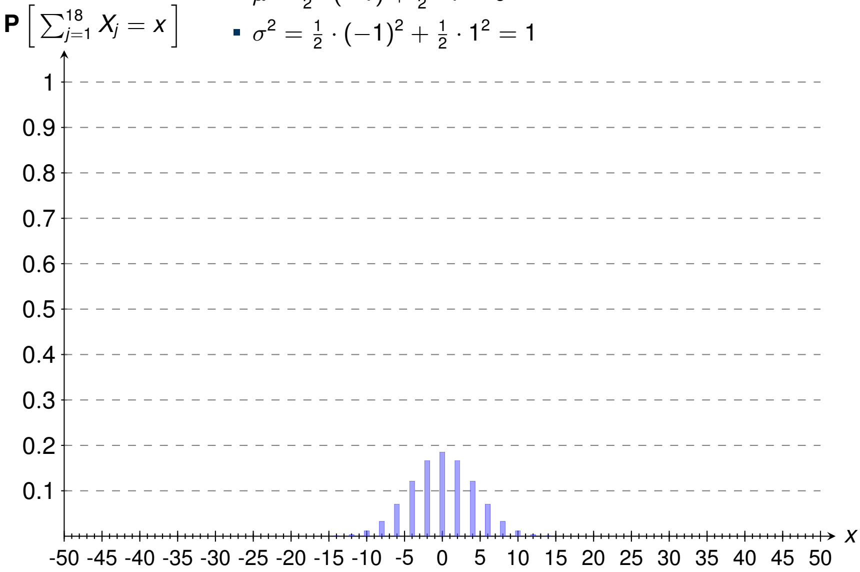

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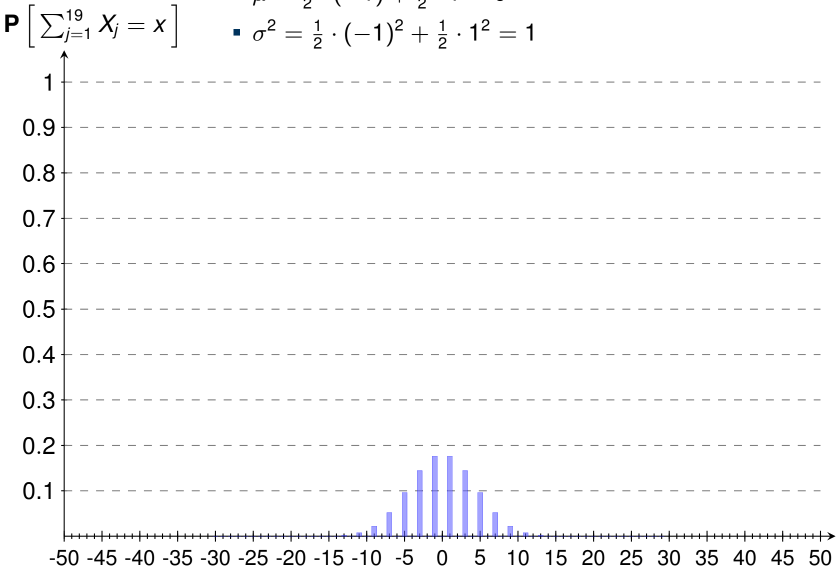

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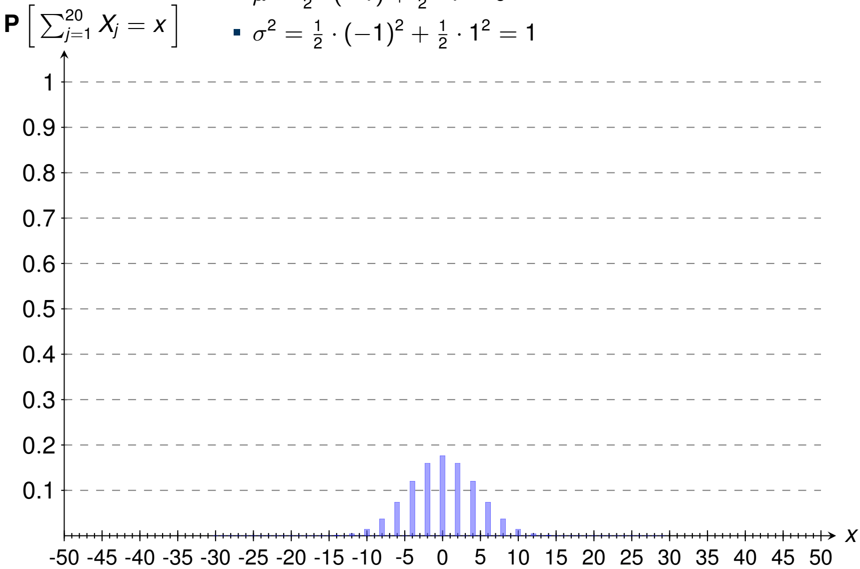

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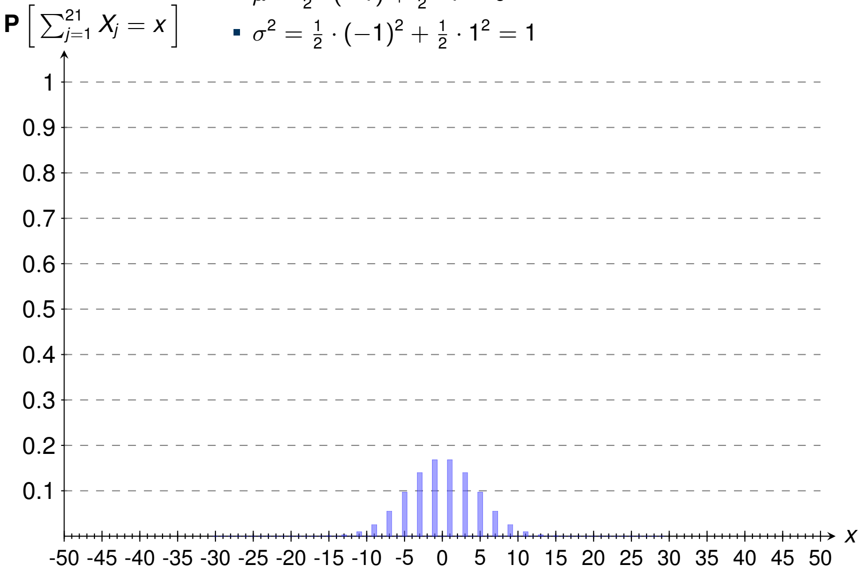

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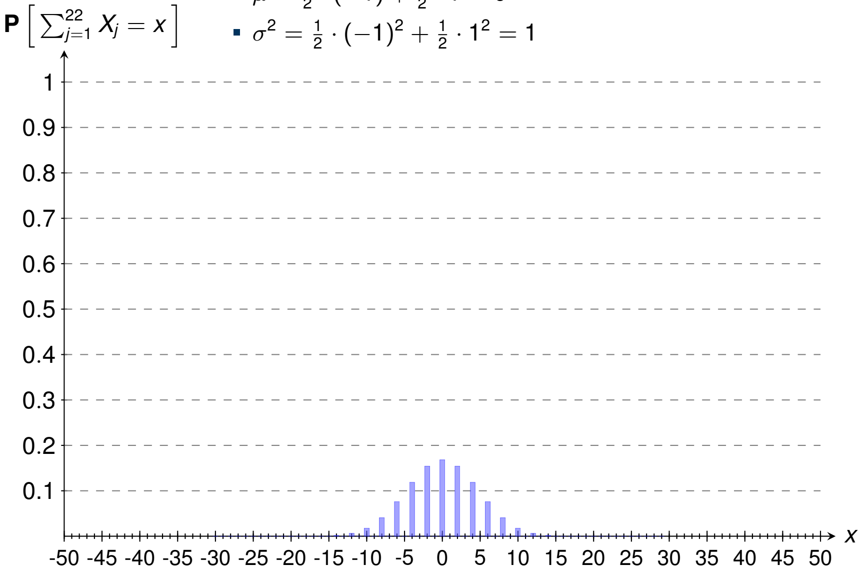

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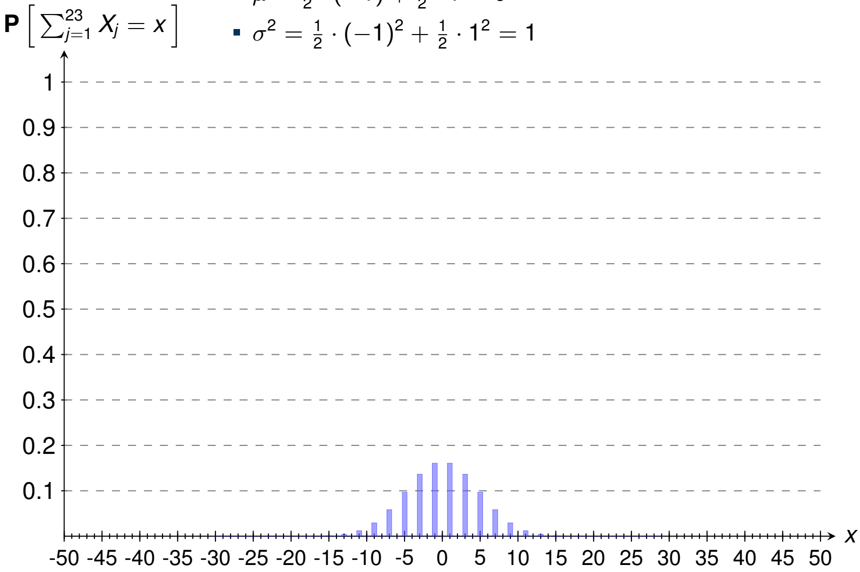

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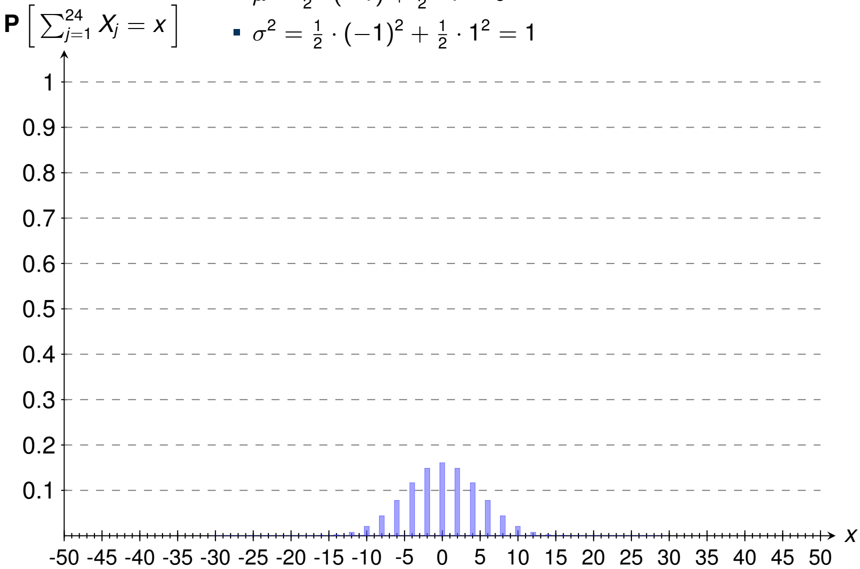

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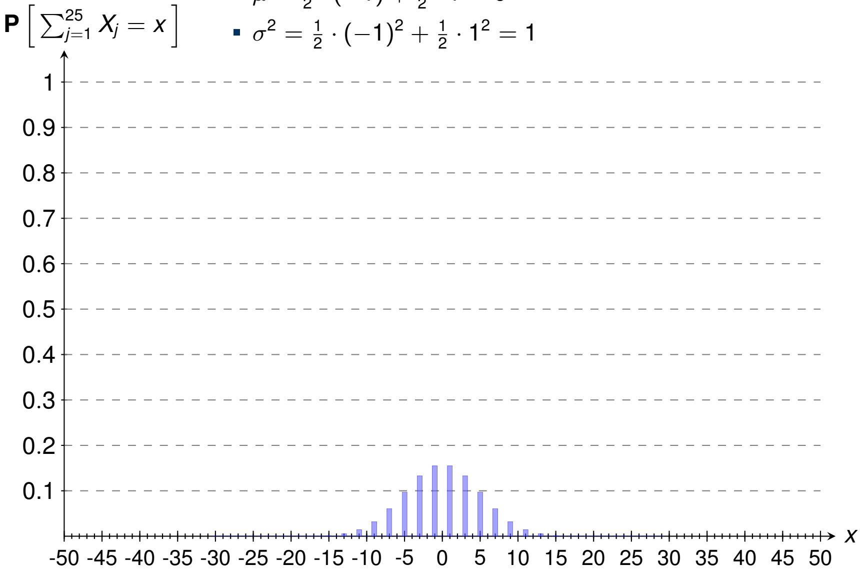

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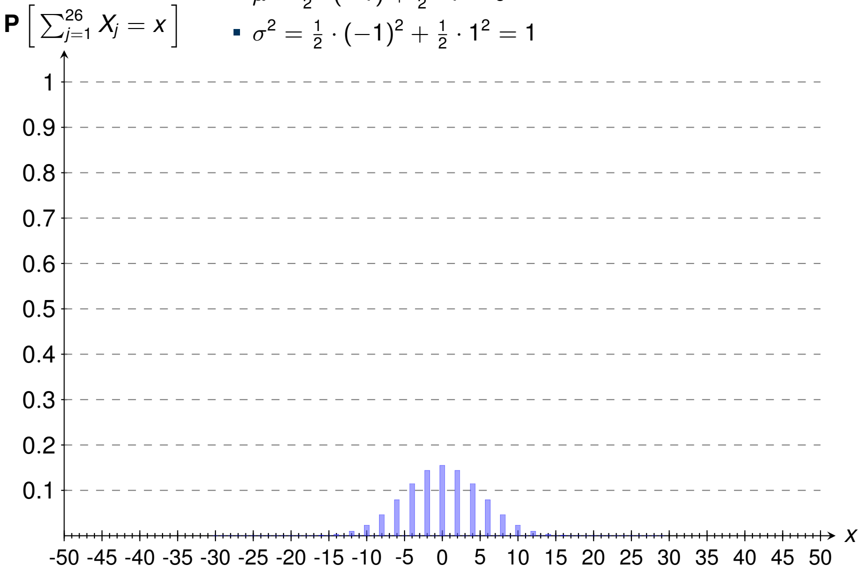

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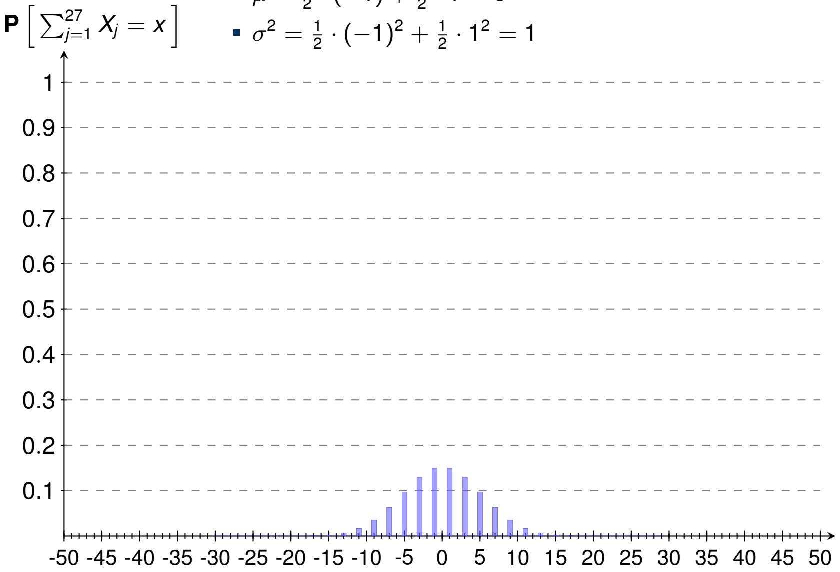

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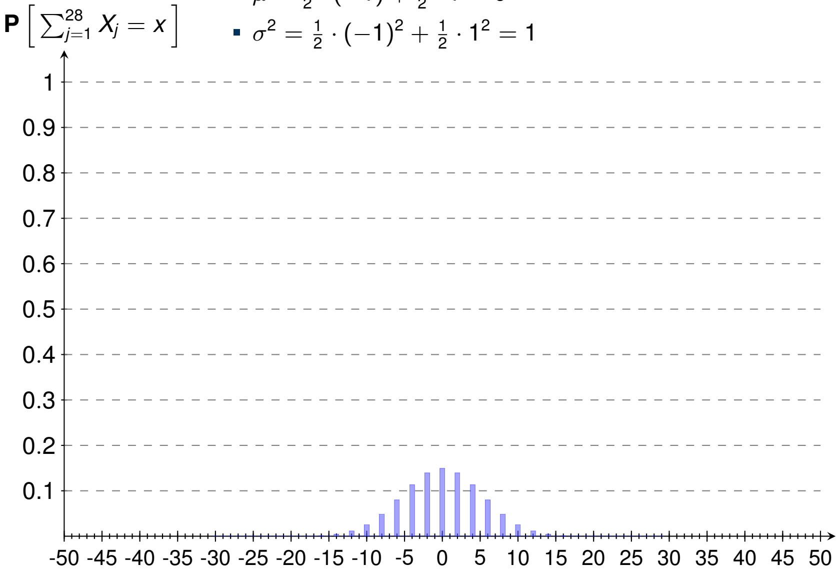

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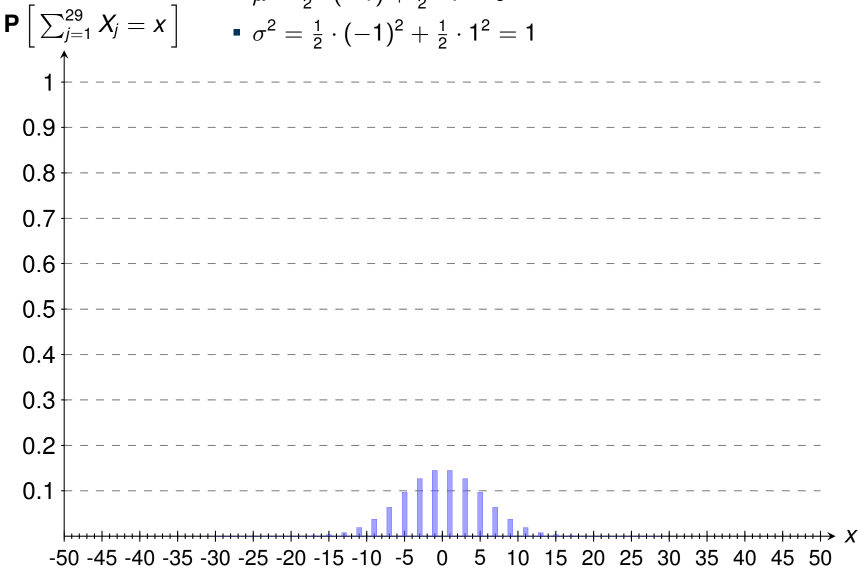

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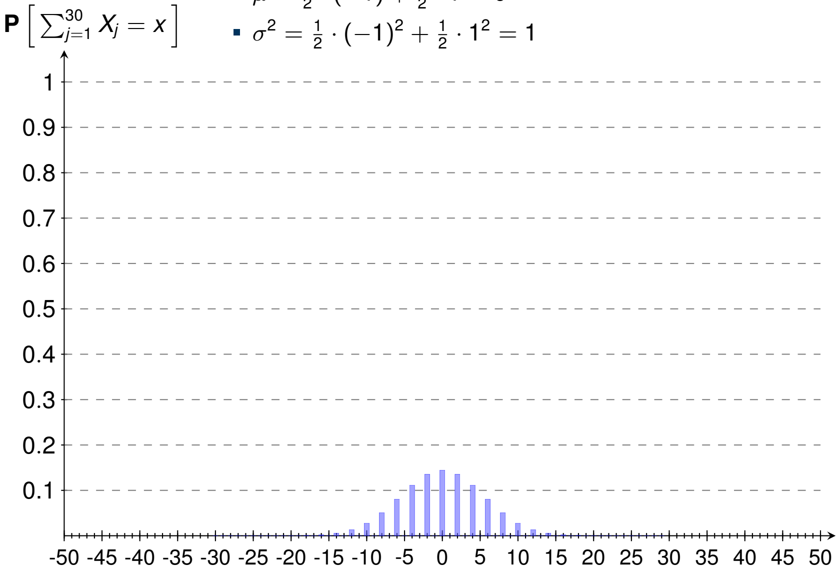

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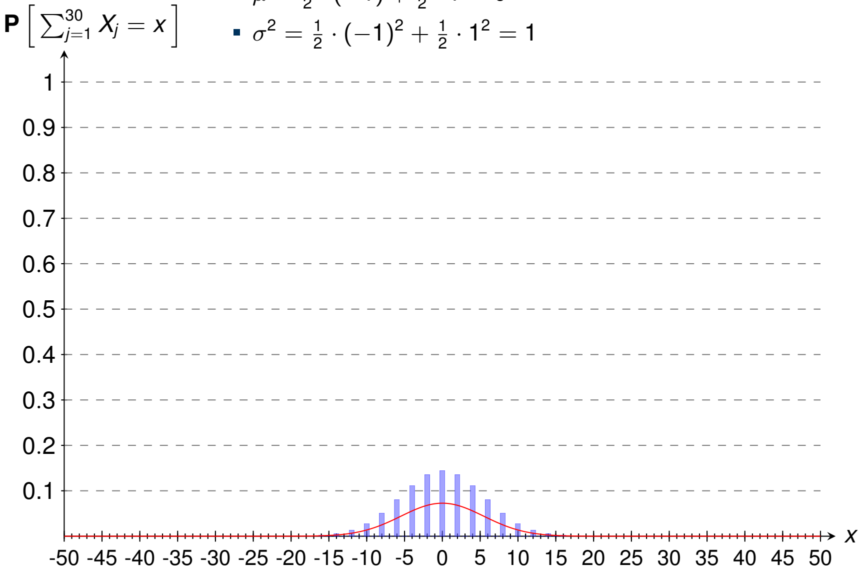

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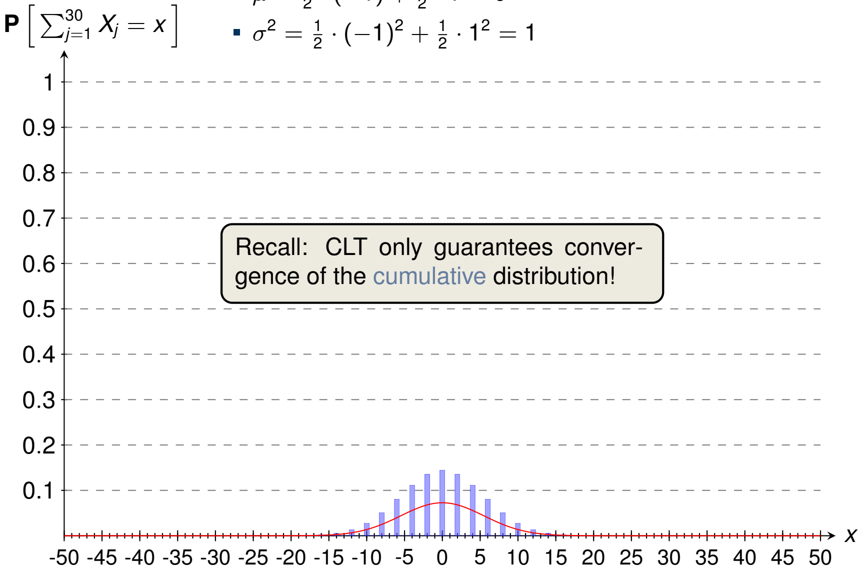

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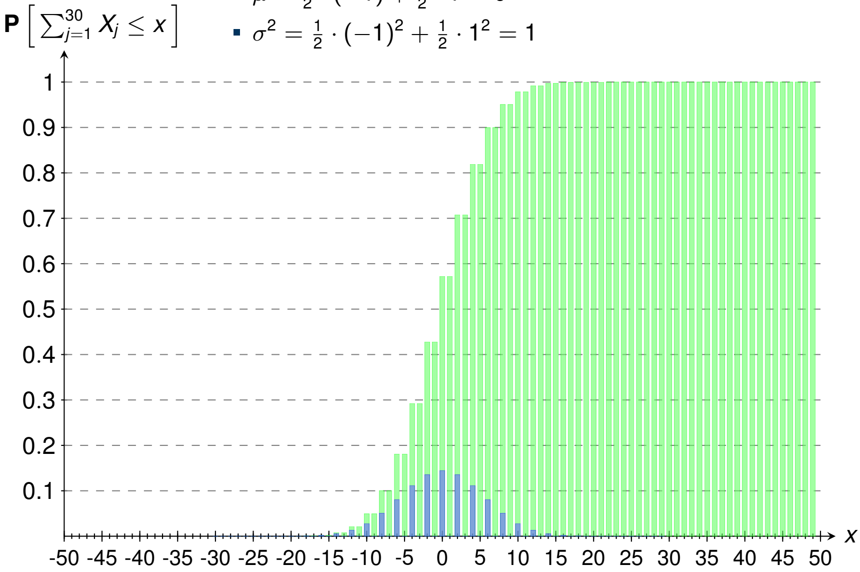

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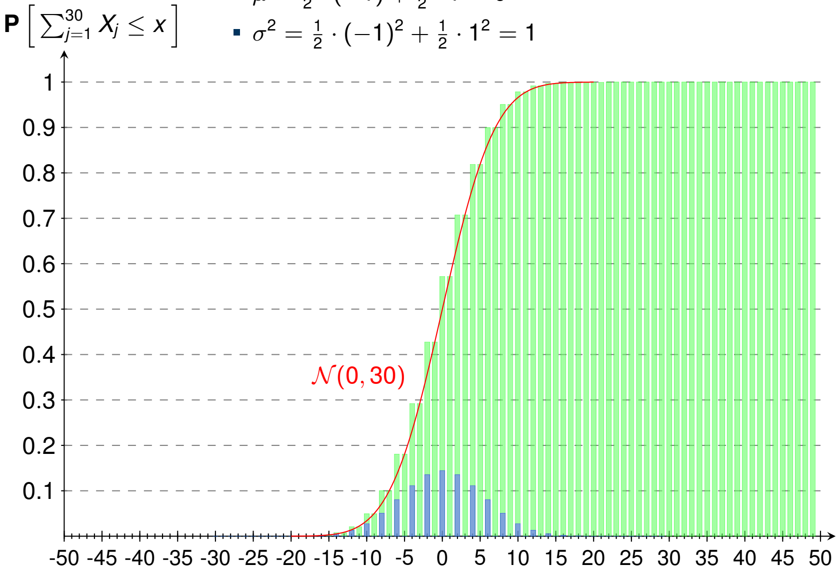

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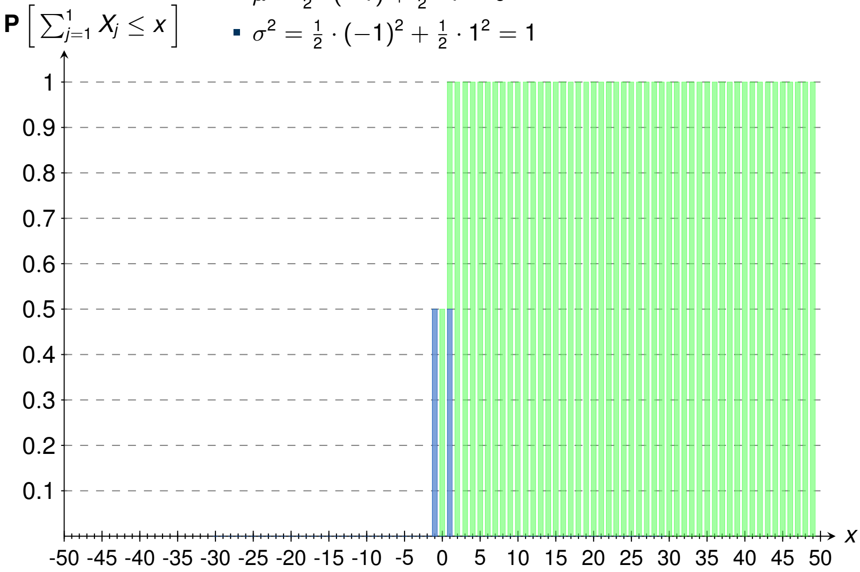

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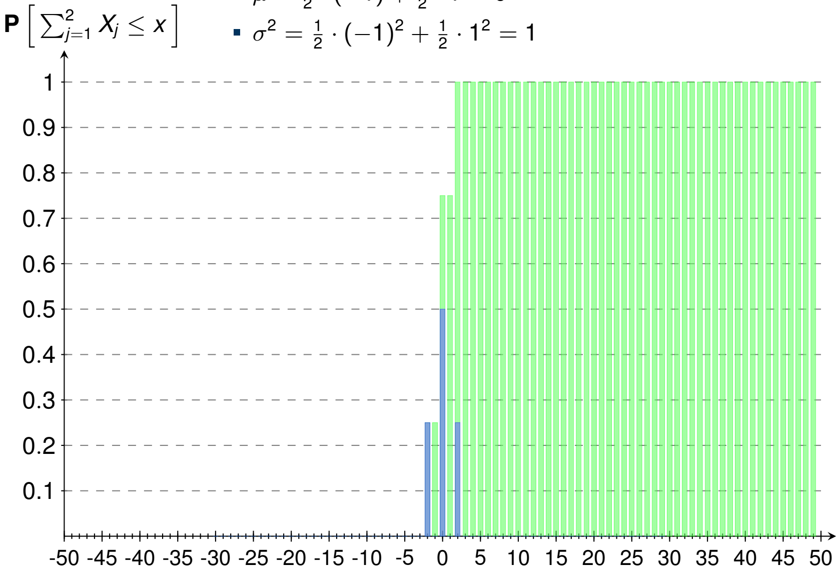

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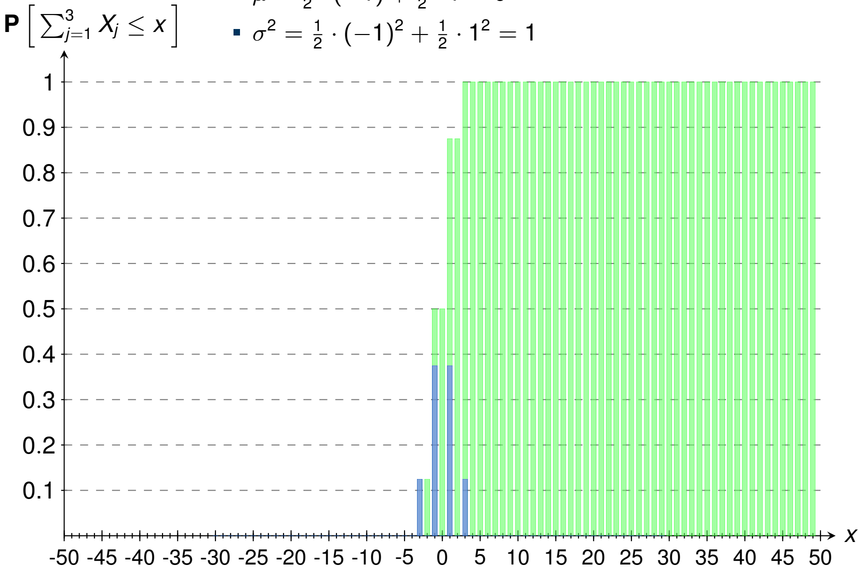

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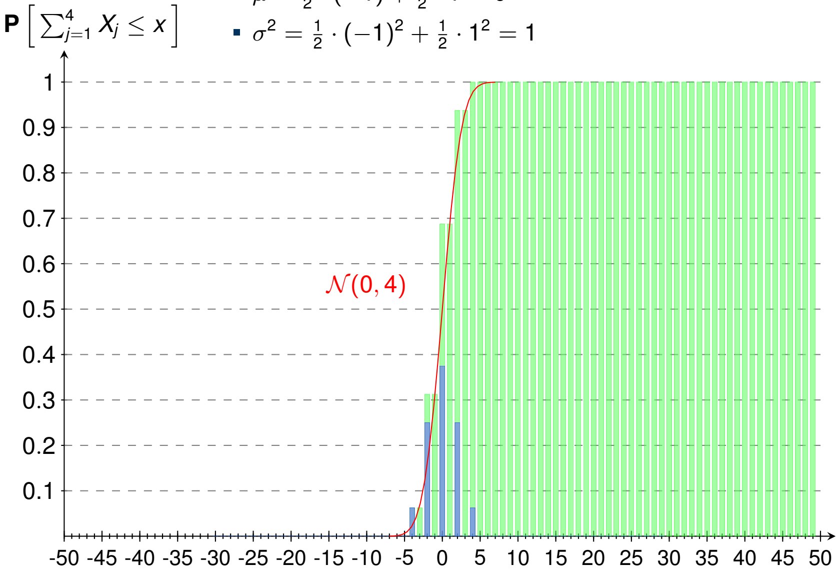

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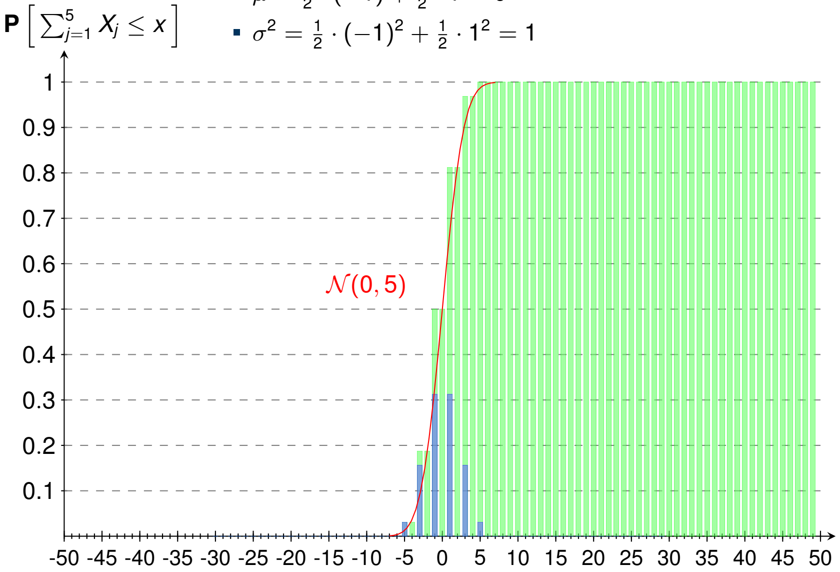

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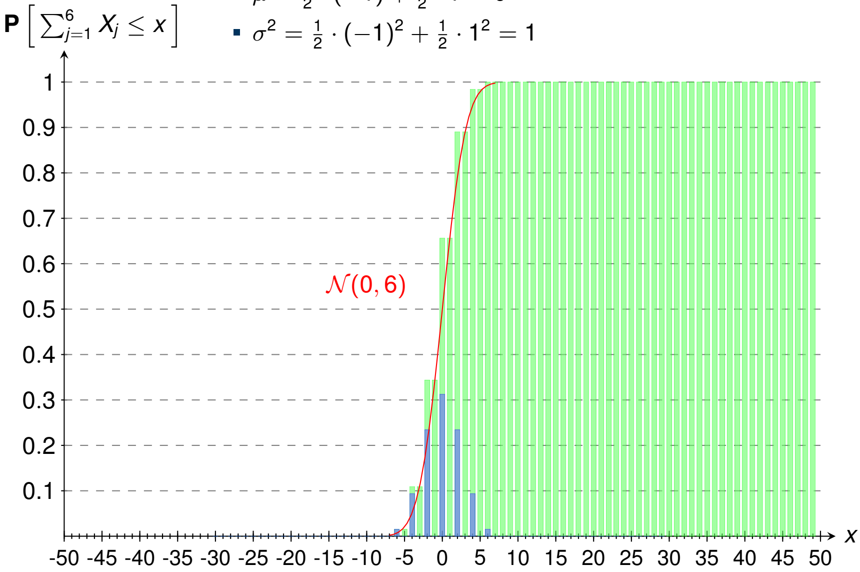

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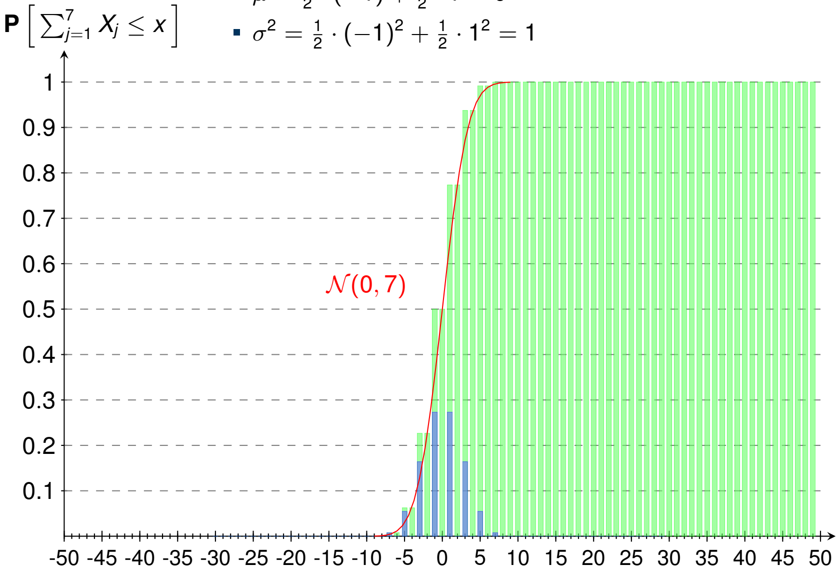

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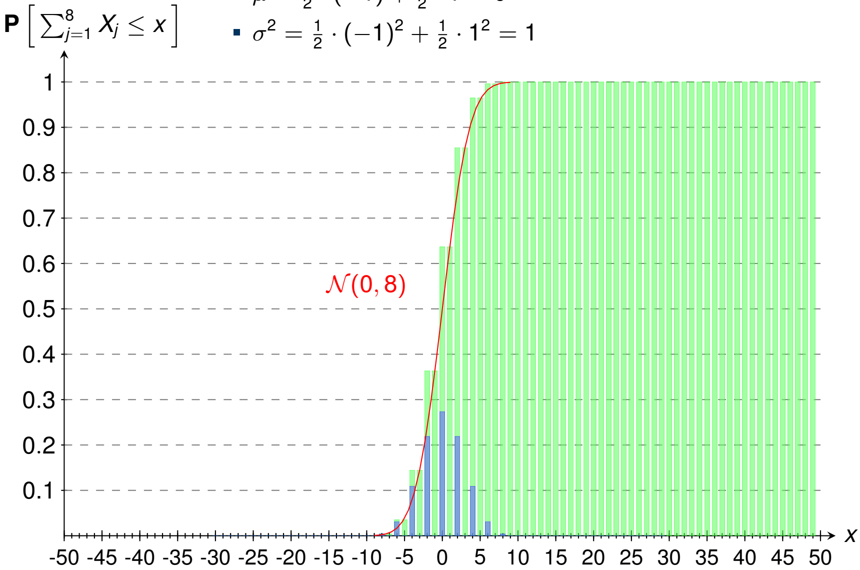

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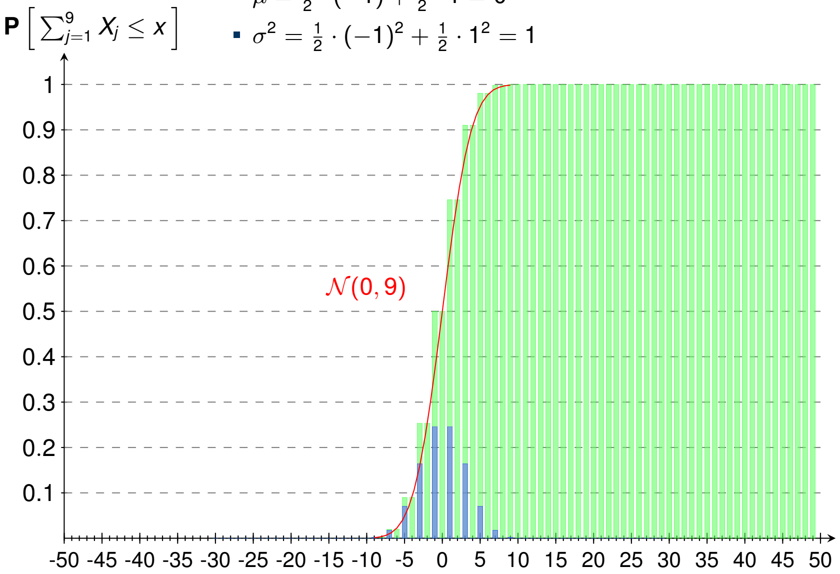

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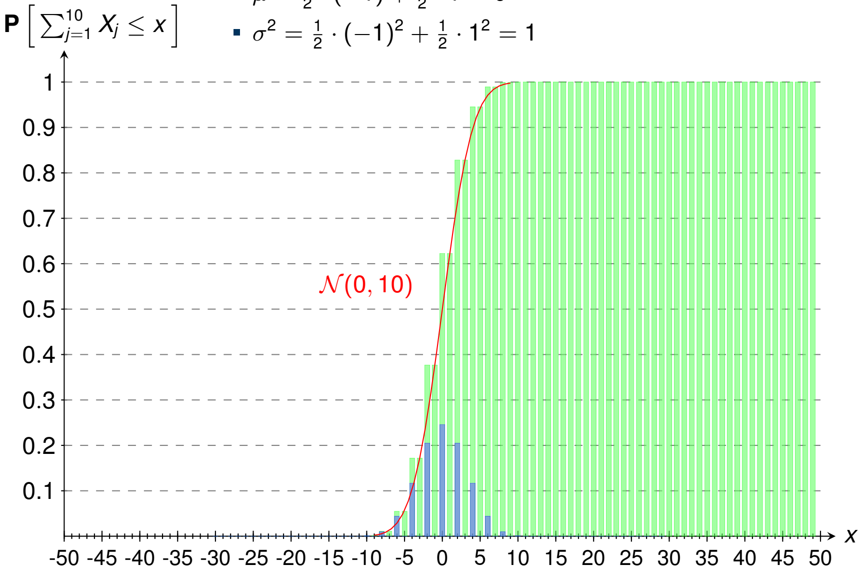

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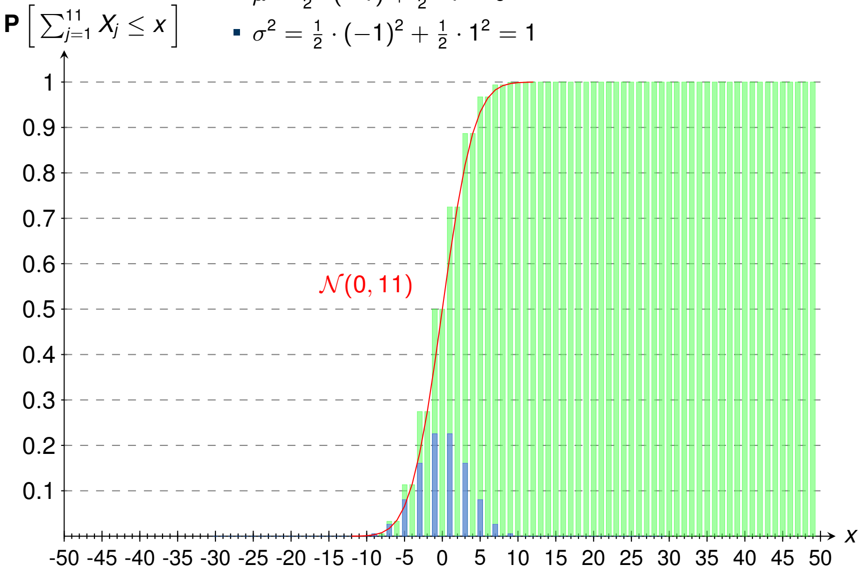

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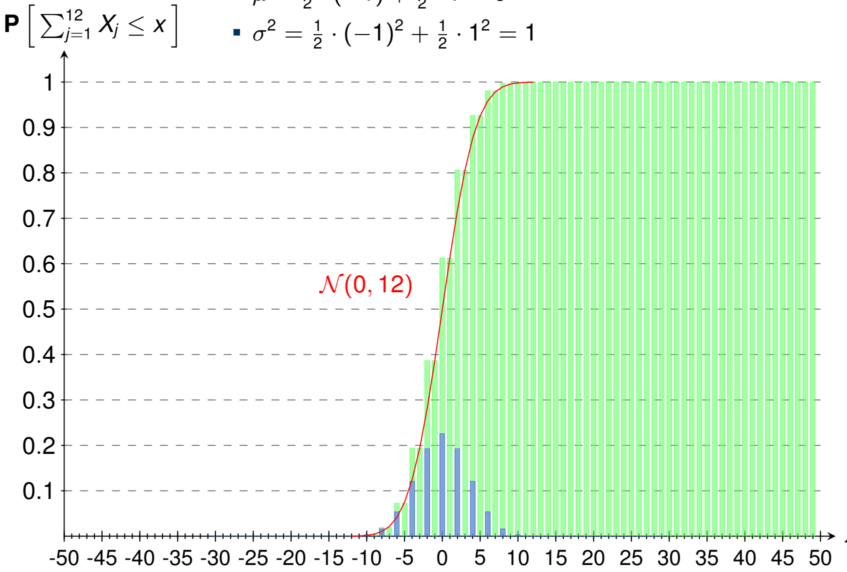

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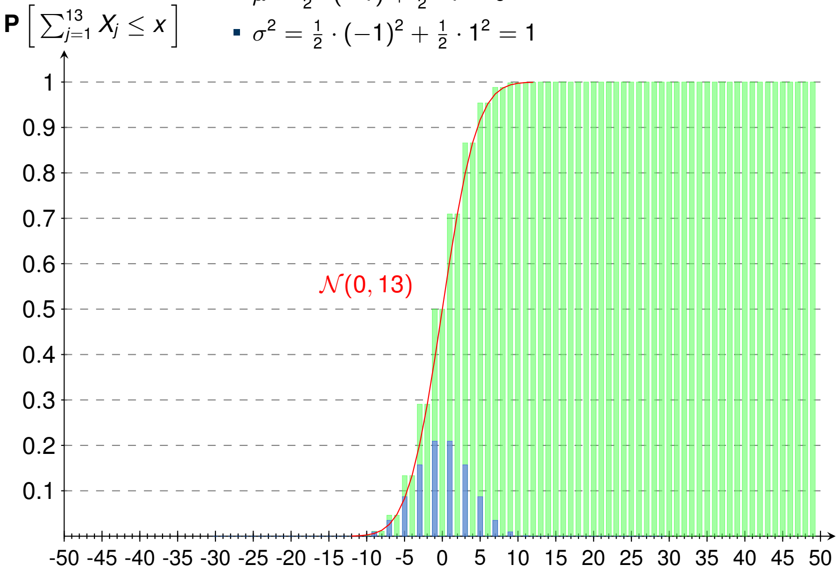

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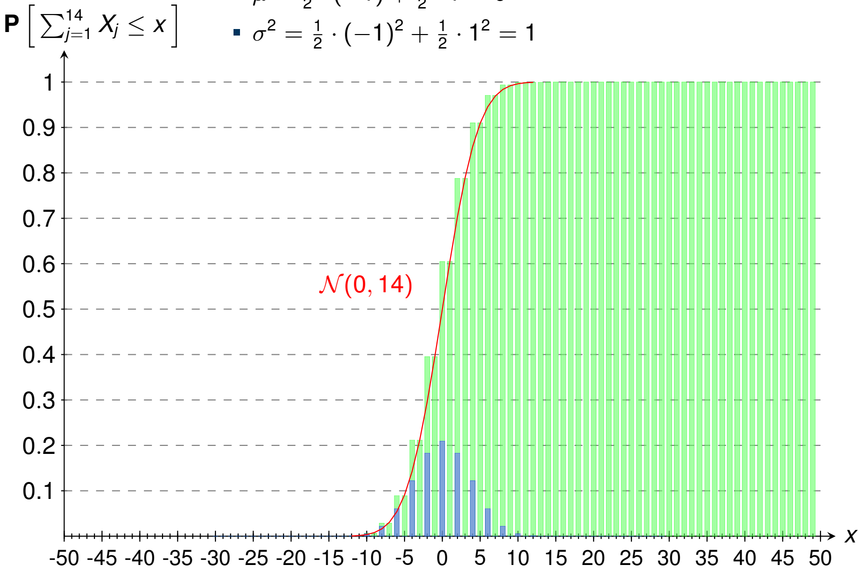

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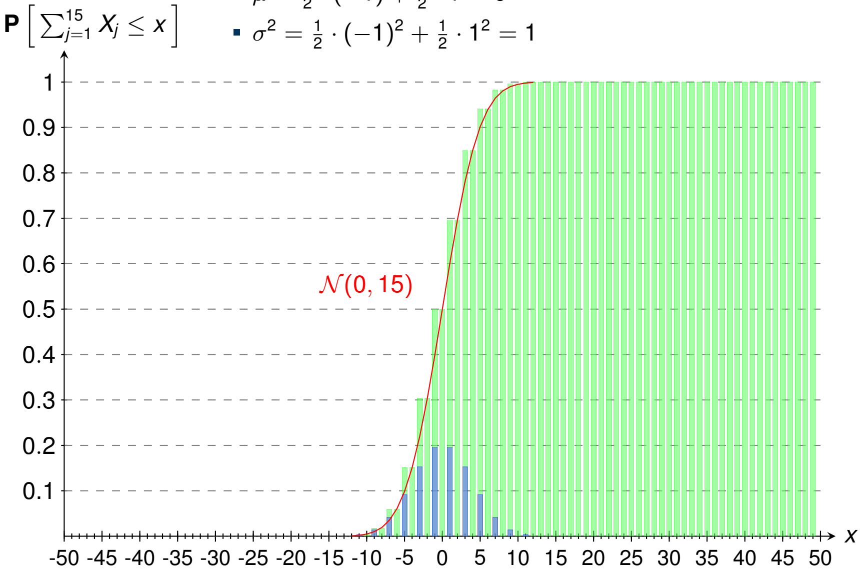

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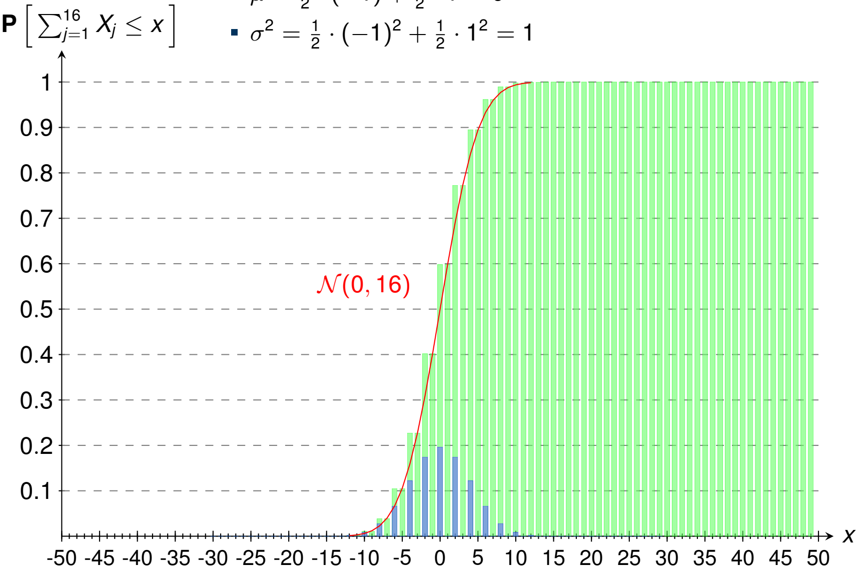

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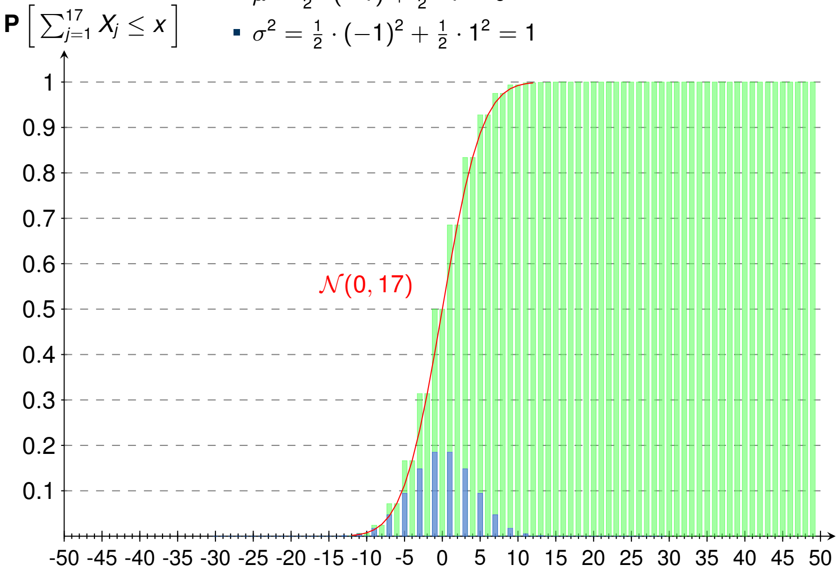

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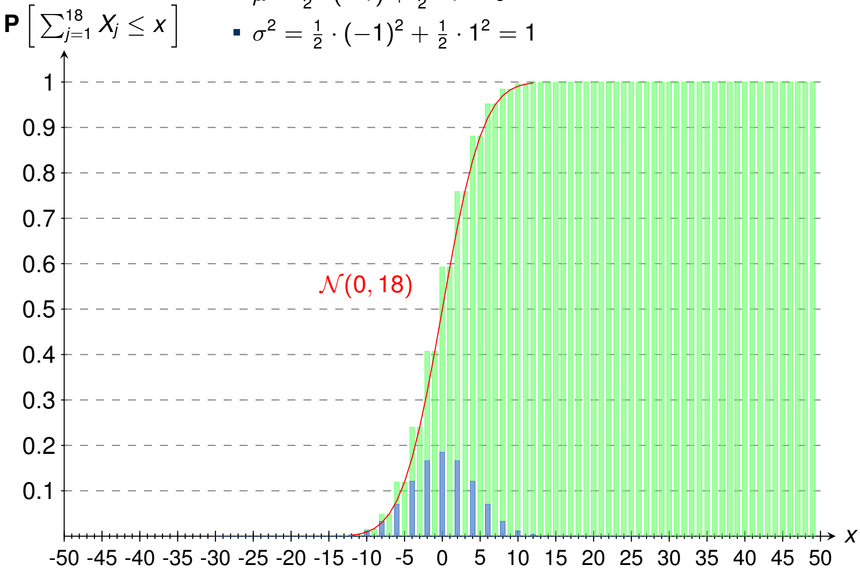

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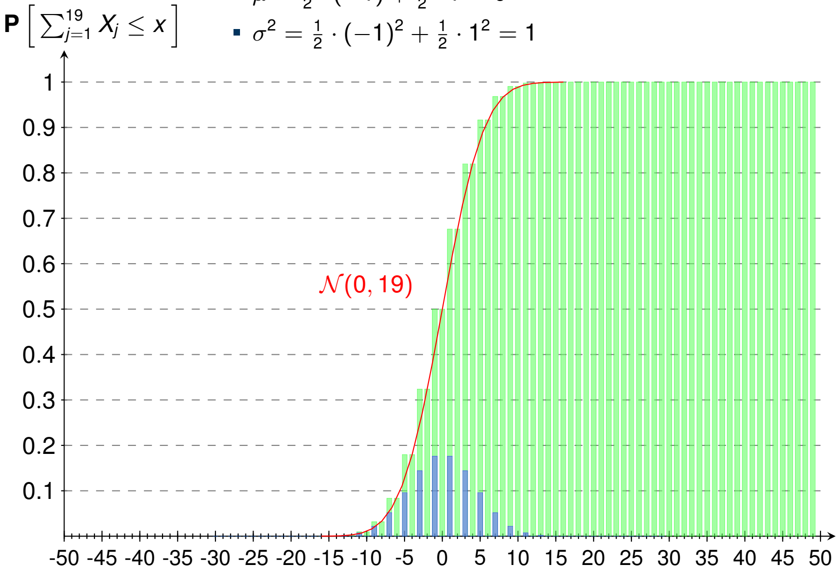

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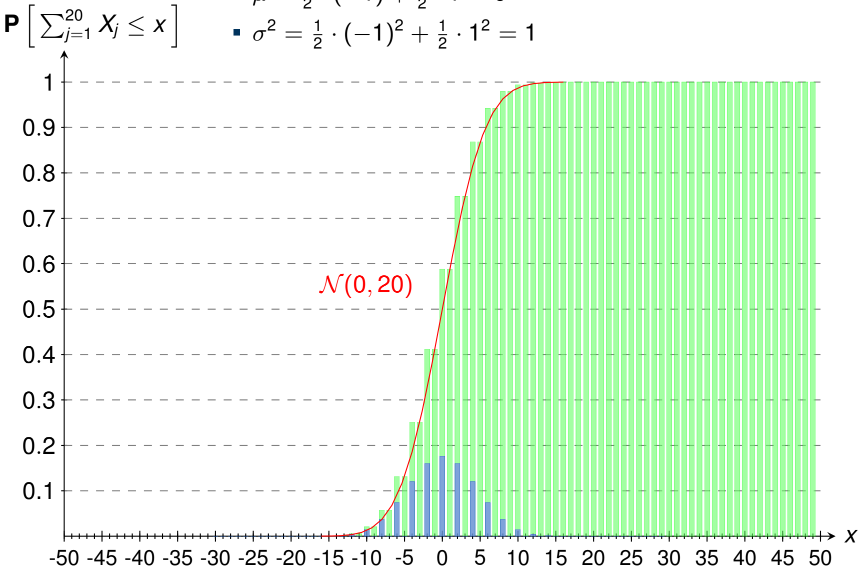

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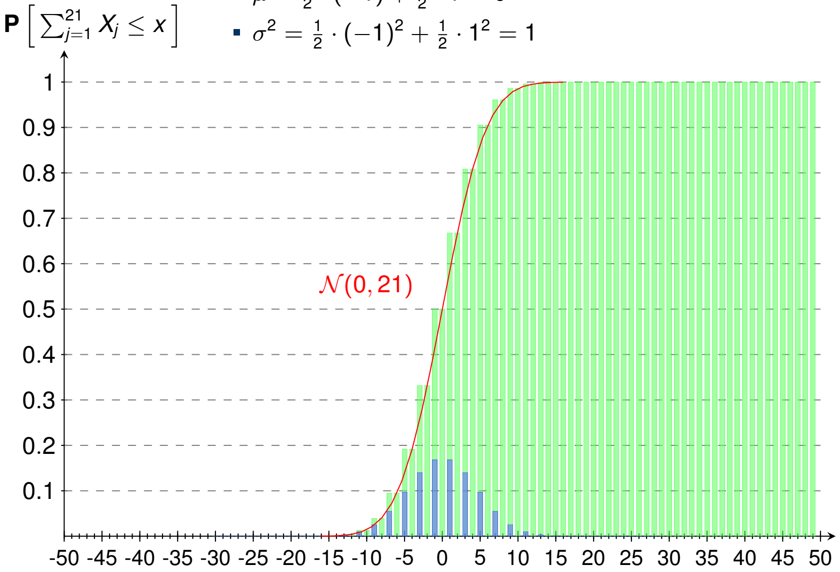

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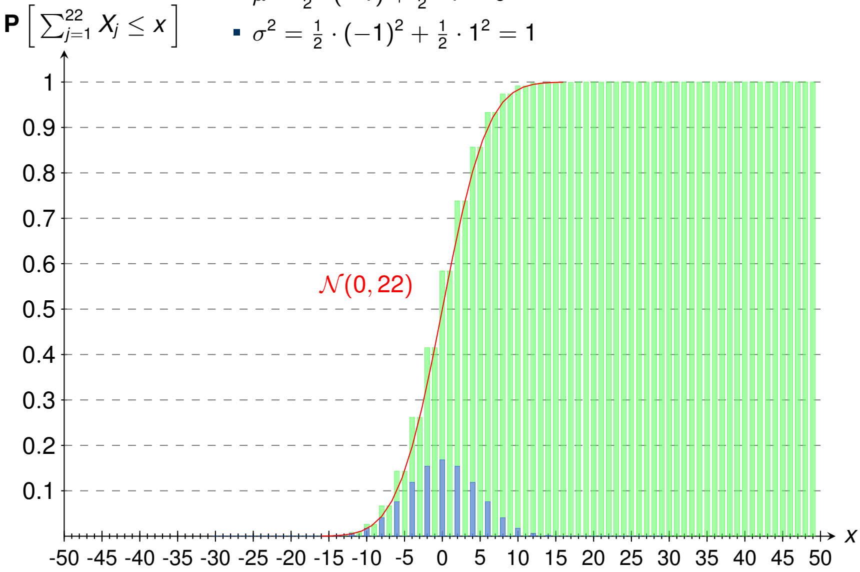

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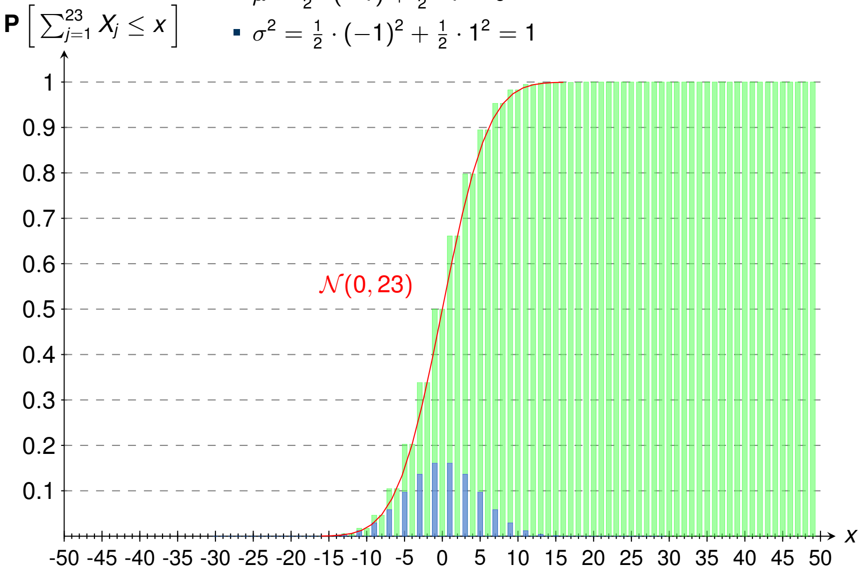

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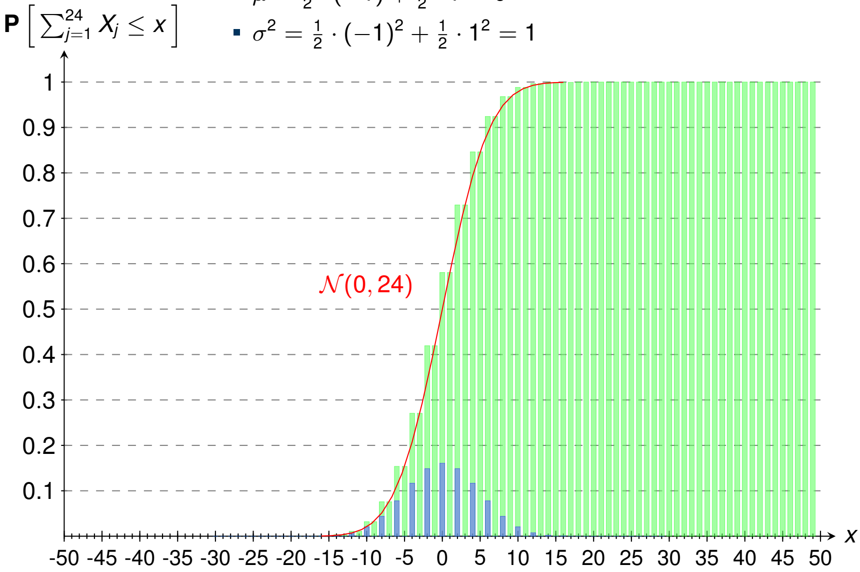

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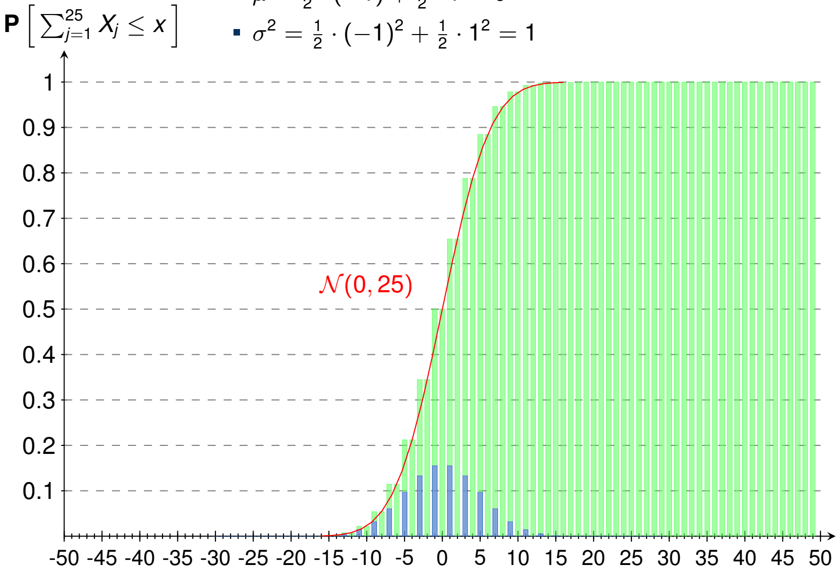

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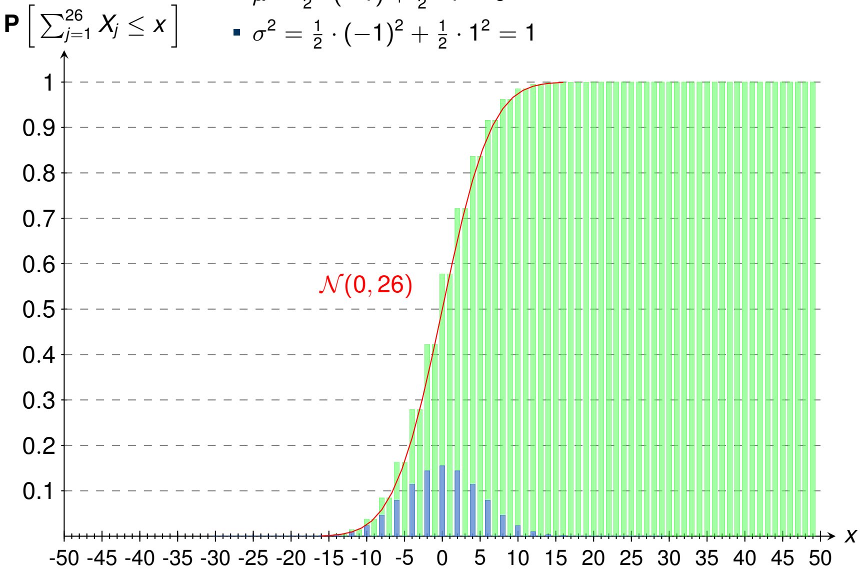

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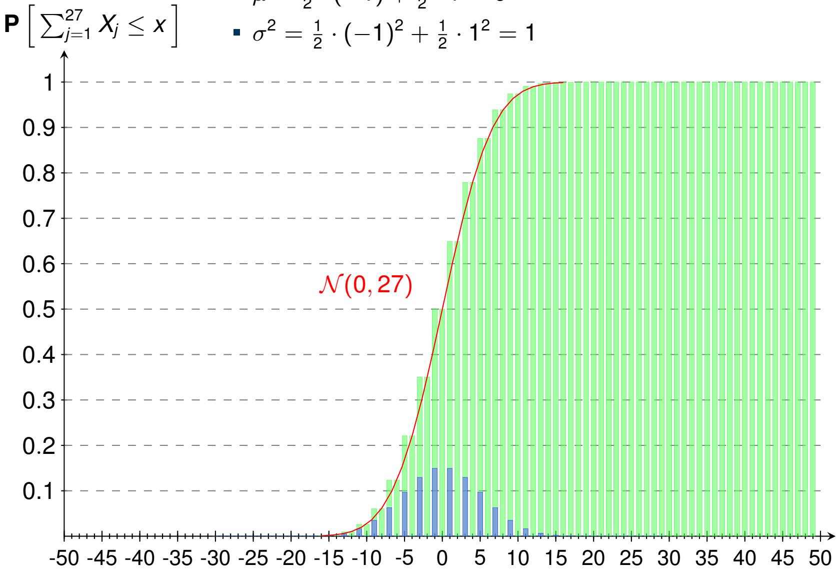

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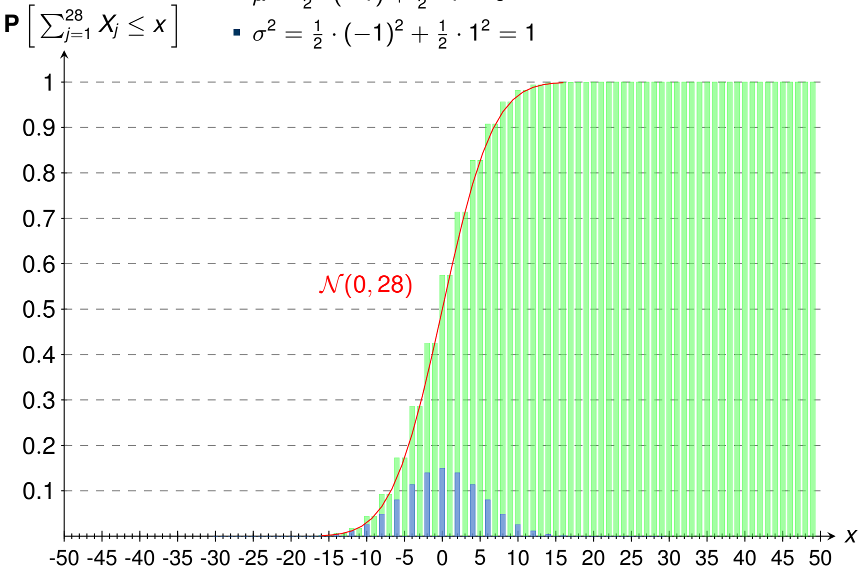

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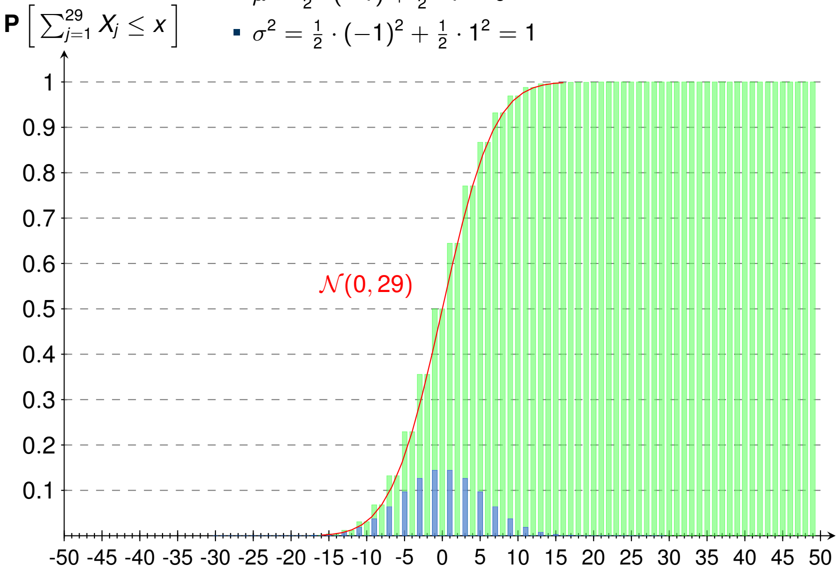

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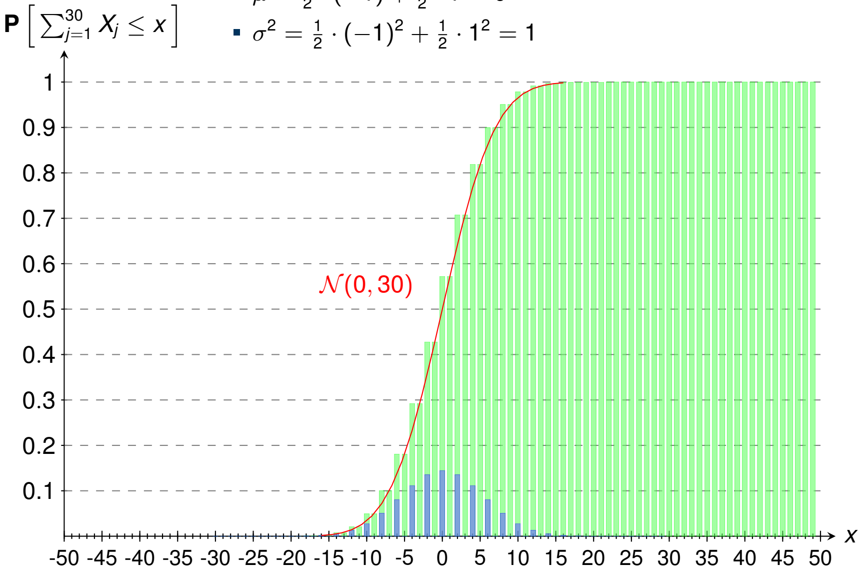

Illustration of CLT with Standardising

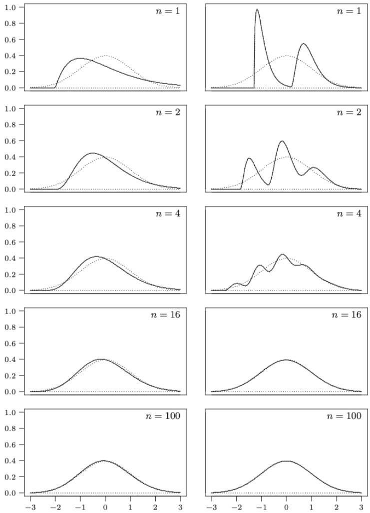

Fig. 14.2. Densities of standardized averages Z_n. Left column: from a gamma density; right column: from a bimodal density. Dotted line: $N(0,1)$ probability density.

Source: Deeking et al., Modern Introduction to Statistics

Outline

Recall: Standard Normal Table

Section 5.4 Normal Random Variables **201**

TABLE 5.1: AREA $\Phi(x)$ UNDER THE STANDARD NORMAL CURVE TO THE LEFT OF X

X	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Source: Ross, Probability 8th ed.

$$Z \sim \mathcal{N}(0,1) \qquad \mathbf{P}[\,Z \leq x\,] = \Phi(x)$$

Normal Approximation of the Binomial Distribution

Example 1

Suppose you are attending a multiple-choice exam of 10 questions and you are completely unprepared. Each question has 4 choices, and you are going to pass the exam if you guess at least 6 correct answers. Use the normal approximation to estimate the probability of passing.

Answer

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True value is 0.0197. Error lies in the discretisation!

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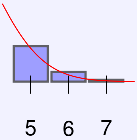

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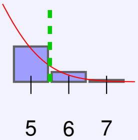
A better approximation is obtained by $\mathbf{P}\left[\sum_{i=1}^{n} X_i \geq 5.5\right] \rightsquigarrow \approx 0.0143$



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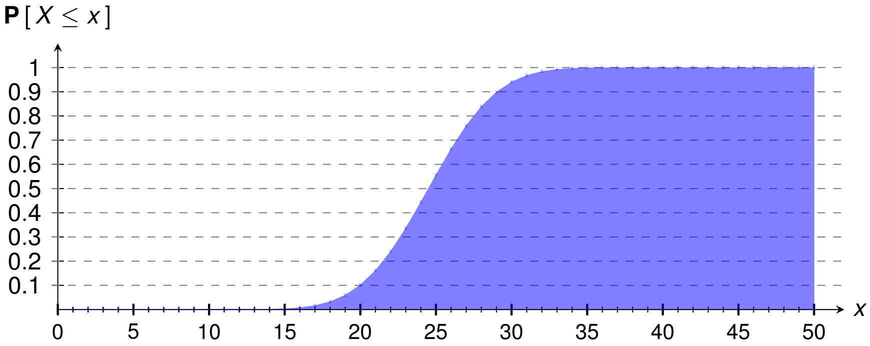
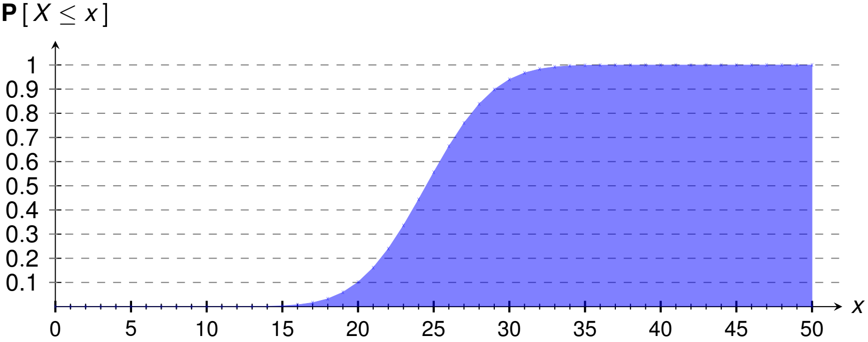
Approximation of the Binomial Distribution

- Let $X \sim Bin(50, 1/2)$



Approximation of the Binomial Distribution

- Let $X \sim Bin(50, 1/2)$
- Hence $\mu = 25$, $\sigma^2 = 50 \cdot 1/4 = 12.5$



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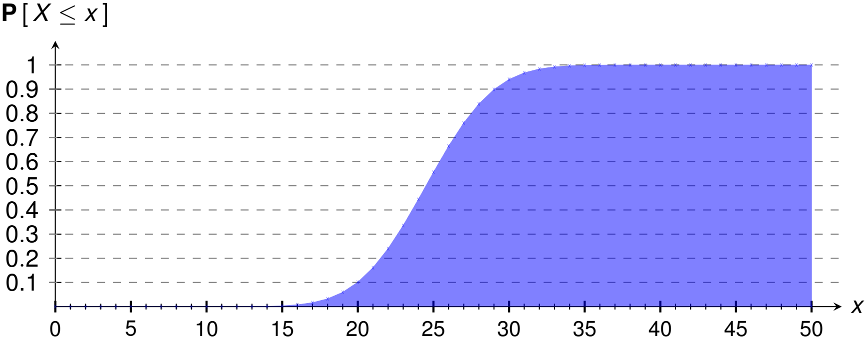

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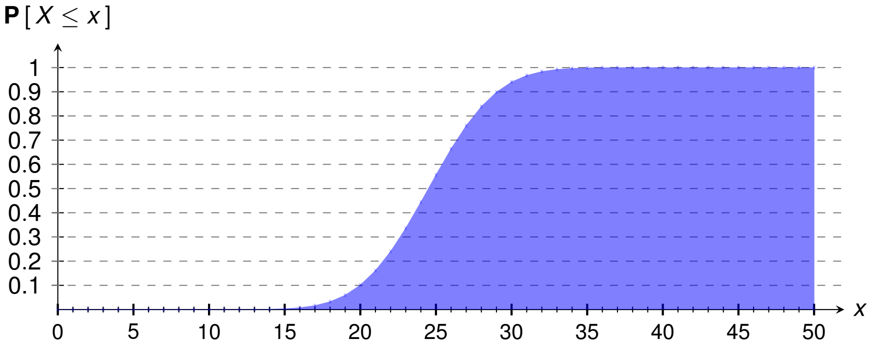

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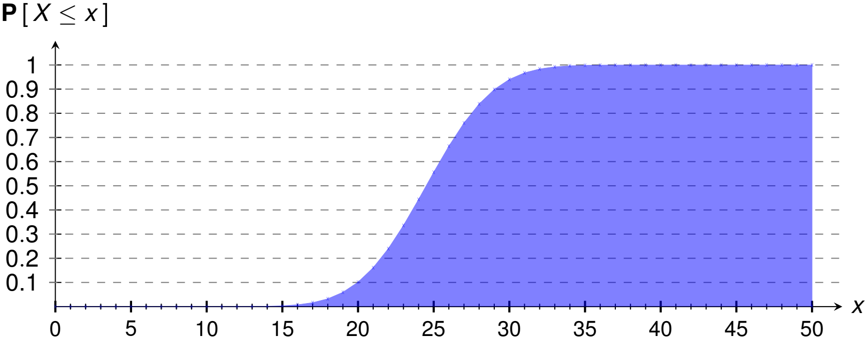

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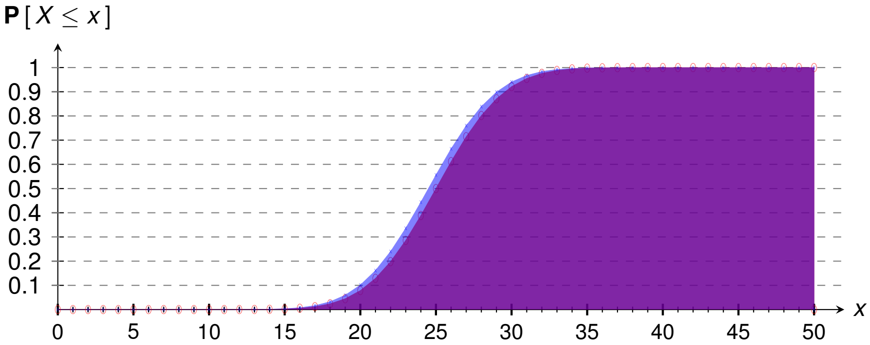

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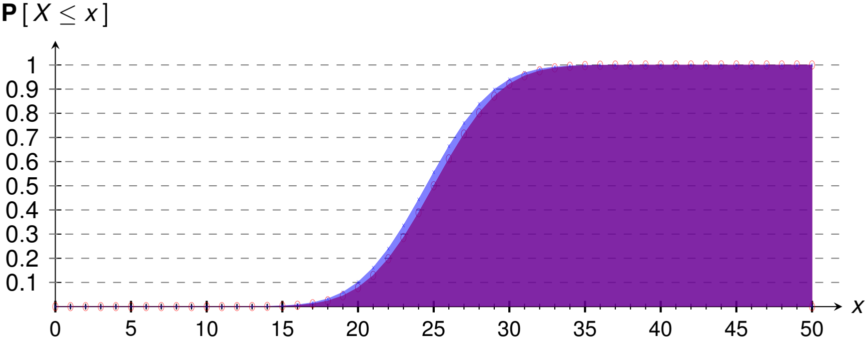
- $\mathbf{P}[X \leq x] \approx \mathbf{P}[Y \leq x + 0.5]$



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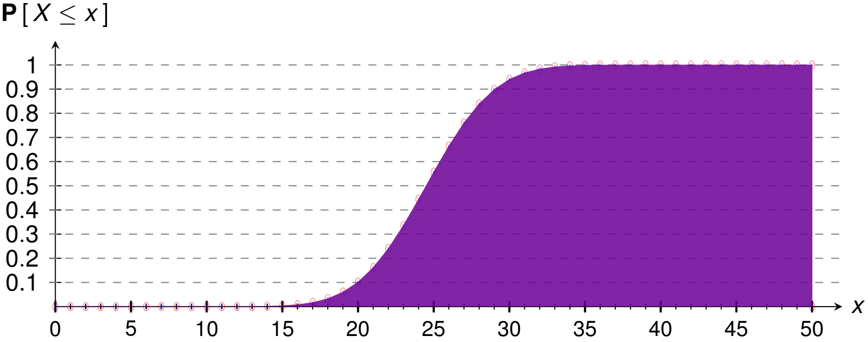

A "Reverse" Application of the CLT

Example 2

Suppose we are sequentially loading one container with packets, whose weights are i.i.d. exponential variables with parameter $\lambda = 1/2$. The container has a capacity of 100 weight units. How many packets can we load so that we meet the capacity threshold with at least .95 probability?

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$\Rightarrow$ Solving the quadratic gives $n \leq 39.6$.

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- Addendum: Following the reasoning from Example 1, $\mathbf{P}\left[\sum_{i=1}^{n} X_i \geq 99.5\right]$ might be a better approximation

A "Reverse" Application of the CLT

Example 2

Suppose we are sequentially loading one container with packets, whose weights are i.i.d. exponential variables with parameter $\lambda = 1/2$. The container has a capacity of 100 weight units. How many packets can we load so that we meet the capacity threshold with at least .95 probability?

Answer

- We have $X_1, X_2, \dots, X_n \sim Exp(1/2)$, where n is unknown.
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- By the CLT,

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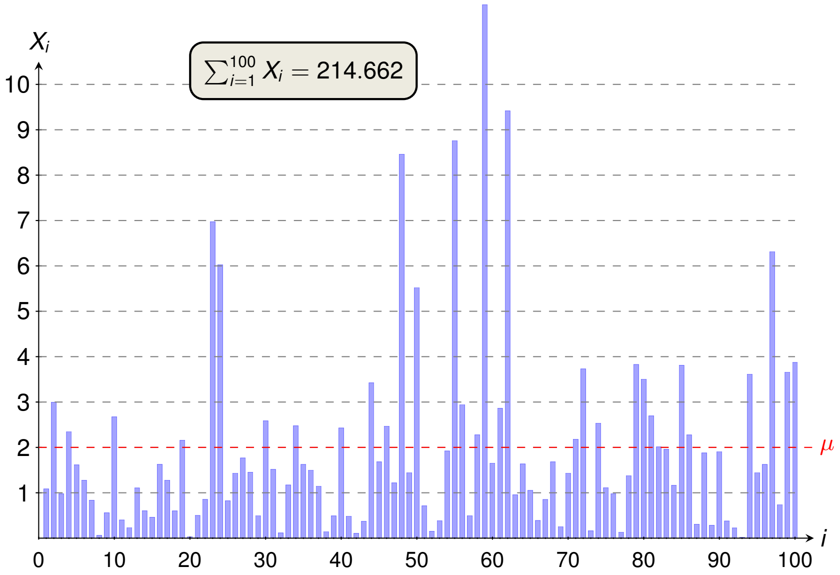
A Sample of 100 Exponential Random Variables $Exp(1/2)$



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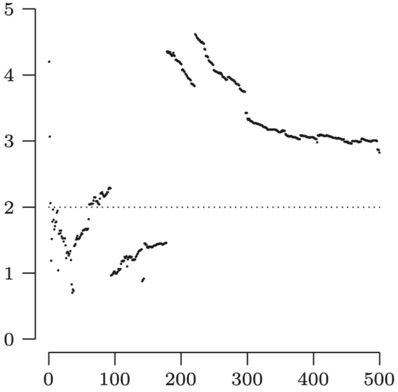
  - In this region, 75 gives a better approximation than 74.5, but for smaller values (e.g., ≤ 63) the ".5-shift" gives significantly better results.

A Distribution whose Average does not converge



$Cau(2,1)$ distribution, Source: Deeking et al., Modern Introduction to Statistics

The Cauchy distribution has "too heavy" tails (no expectation), in particular the average does not converge.

Outline

Recap: Weak Law of Large Numbers

Central Limit Theorem

Illustrations

Examples

Bonus Material (non-examinable)

Towards a Proof of CLT: Moment Generating Functions

Moment-Generating Function

The moment-generating function of a random variable X is

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Proof of 2: (Proof of 1 is quite non-trivial!)

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- Differentiating (details ommitted here, see book by Ross) shows $L(0) = 0, L'(0) = \mu = 0$ and $L''(0) = \mathbf{E}\left[X^2 \right] = 1$.

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- To prove the theorem, we must show that

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Proof Sketch (cntd):

- To prove the theorem, we must show that

$$\lim_{n\to\infty} \left(M\left(\frac{t}{\sqrt{n}}\right)\right)^n \to e^{t^2/2}$$

This is the moment generating function of $N(0,1)$.

- We take logarithms on both sides and obtain

$$\lim_{n\to\infty} \frac{L(t/\sqrt{n})}{n^{-1}} = \lim_{n\to\infty} \frac{-L'(t/\sqrt{n})n^{-3/2}t}{-2n^{-2}}$$

Using L'Hopital's rule.

$$= \lim_{n\to\infty} \frac{-L'(t/\sqrt{n})t}{2n^{-1/2}}$$

Using L'Hopital's rule (again)

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&= \lim_{n\to\infty}\frac{-L'(t/\sqrt{n})t}{2n^{-1/2}} \\
&= \lim_{n\to\infty}\frac{-L''(t/\sqrt{n})n^{3/2}t^2}{-2n^{-3/2}} \\
&= \lim_{n\to\infty}\left[-L''(t/\sqrt{n})n^{3/2}\cdot\frac{t^2}{2}\right]
\end{aligned}$$

Using L'Hopital's rule.

Using L'Hopital's rule (again)

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We have $L''(0) = 1$!

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&= \frac{t^2}{2}.
\end{aligned}$$

Using L'Hopital's rule.

Using L'Hopital's rule (again)

We have $L''(0) = 1$!

We proved that the MGF of Z_n converges to that one of $\mathcal{N}(0,1)$.