Foundations of Computer Science
Lecture 7:
Dictionaries and Functional Arrays

Anil Madhavapeddy

21st October 2022

Warmup

```
# type 'a tree =
 | Br of 'a * 'a tree * 'a tree
 | ??
```

What's the missing definition here to make a binary tree?

Warmup

```
# type 'a tree =
  | Br of 'a * 'a tree * 'a tree
  | Lf
```

Warmup

```
# type 'a tree =
  | Br of 'a * 'a tree * 'a tree
  | Lf
```

What is the term when **'a** is present in the type definition?

Warmup

```
# type 'a tree =
 | Br of 'a * 'a tree * 'a tree
 | Lf
```

What is the term when **'a** is present in the type definition?

polymorphic

Warmup

```
# type 'a tree =
 | Br of 'a * 'a tree * 'a tree
 | Lf
```

What is the term when **'a** is present in the type definition?

polymorphic

What is the term when the type definition refers to itself?

Warmup

```
# type 'a tree =
 | Br of 'a * 'a tree * 'a tree
 | Lf
```

What is the term when **'a** is present in the type definition?

polymorphic

What is the term when the type definition refers to itself?

recursive

Warmup

```
# type 'a option =
 | None
 | Some of ‘a

# exception Not_found
```

Why use option types *vs* raising exceptions?

Warmup

```
# type 'a option =
 | None
 | Some of ‘a

# exception Not_found
```

Why use option types *vs* raising exceptions?

```
val change_exn : int list -> int -> int list
val change_opt : int list -> int -> int list option
```

Every call to change_opt must check the option

```
match change_opt with
| None -> … (* error case *)
| Some ch -> … (* success case *)
```

Dictionaries

- A dictionary attaches **values** to identifiers (known as **keys**).
- Define the **operations** we want over the dictionary:
  - **`lookup`** : find an item in the dictionary
  - **`update`** / `insert` : replace / store an item in the dictionary
  - `delete` : remove an item from the dictionary
  - `empty` : the null dictionary with no keys
  - **`Missing`** : exception for errors in lookup and delete

Implementing a dictionary

- Simplest representation for a dictionary is an **association list** (a list of key/value tuples).

```
# exception Missing
exception Missing
```

Implementing a dictionary

- Simplest representation for a dictionary is an **association list** (a list of key/value tuples).

```
# exception Missing
exception Missing

# let rec lookup = function
  | [], a -> raise Missing
  | (x, y) :: pairs, a ->
      if a = x then
        y
      else
        lookup (pairs, a)
val lookup : ('a * 'b) list * 'a -> 'b = <fun>
```

Implementing a dictionary

- Simplest representation for a dictionary is an **association list** (a list of key/value tuples).

```
# exception Missing
exception Missing

# let rec lookup = function
  | [], a -> raise Missing
  | (x, y) :: pairs, a ->
      if a = x then
        y
      else
        lookup (pairs, a)
val lookup : ('a * 'b) list * 'a -> 'b = <fun>

# let update (l, b, y) = (b, y) :: l
val update : ('a * 'b) list * 'a * 'b -> ('a * 'b) list = <fun>
```

Implementing a dictionary

- Simplest representation for a dictionary is an **association list** (a list of key/value tuples).

```
# exception Missing
exception Missing

# let rec lookup = function
   | [], a -> raise Missing
   | (x, y) :: pairs, a ->
       if a = x then
         y
       else
         lookup (pairs, a)
val lookup : ('a * 'b) list * 'a -> 'b = <fun>

# let update (l, b, y) = (b, y) :: l
val update : ('a * 'b) list * 'a * 'b -> ('a * 'b) list = <fun>
```

Lookup is O(n)

Implementing a dictionary

- Simplest representation for a dictionary is an **association list** (a list of key/value tuples).

```
# exception Missing
exception Missing

# let rec lookup = function
  | [], a -> raise Missing
  | (x, y) :: pairs, a ->
      if a = x then
        y
      else
        lookup (pairs, a)
val lookup : ('a * 'b) list * 'a -> 'b = <fun>

# let update (l, b, y) = (b, y) :: l
val update : ('a * 'b) list * 'a * 'b -> ('a * 'b) list = <fun>
```

Lookup is O(n)

Update is O(1)

Implementing a dictionary

- Simplest representation for a dictionary is an **association list** (a list of key/value tuples).

```
# exception Missing
exception Missing

# let rec lookup = function
    | [], a -> raise Missing
    | (x, y) :: pairs, a ->
        if a = x then
          y
        else
          lookup (pairs, a)
val lookup : ('a * 'b) list * 'a -> 'b = <fun>

# let update (l, b, y) = (b, y) :: l
val update : ('a * 'b) list * 'a * 'b -> ('a * 'b) list = <fun>
```

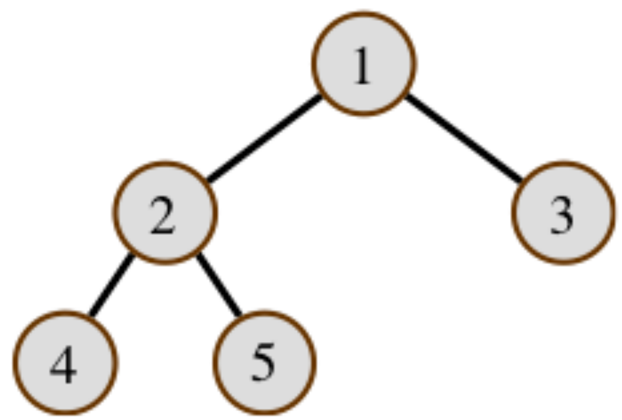
Lookup is O(n)

Update is O(1)

But what is the space usage?

Binary Search Trees

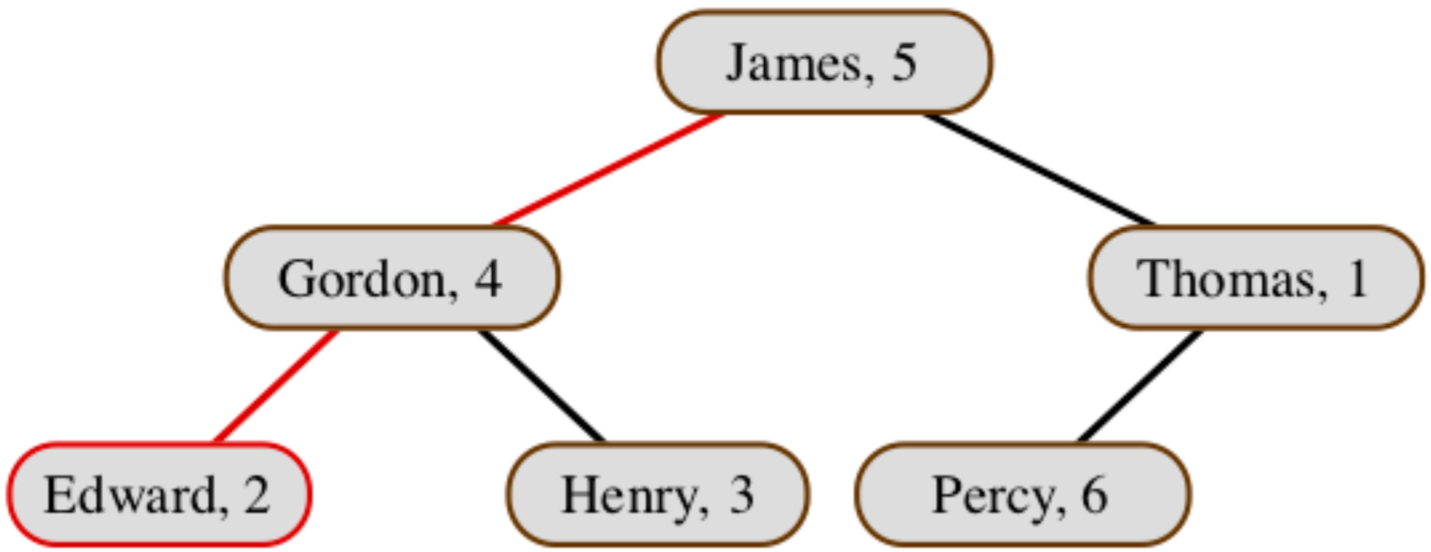
- Use **binary trees** as a more efficient representation than a list to get a better lookup complexity.




```
# type 'a tree =
    Lf
  | Br of 'a * 'a tree * 'a tree
```

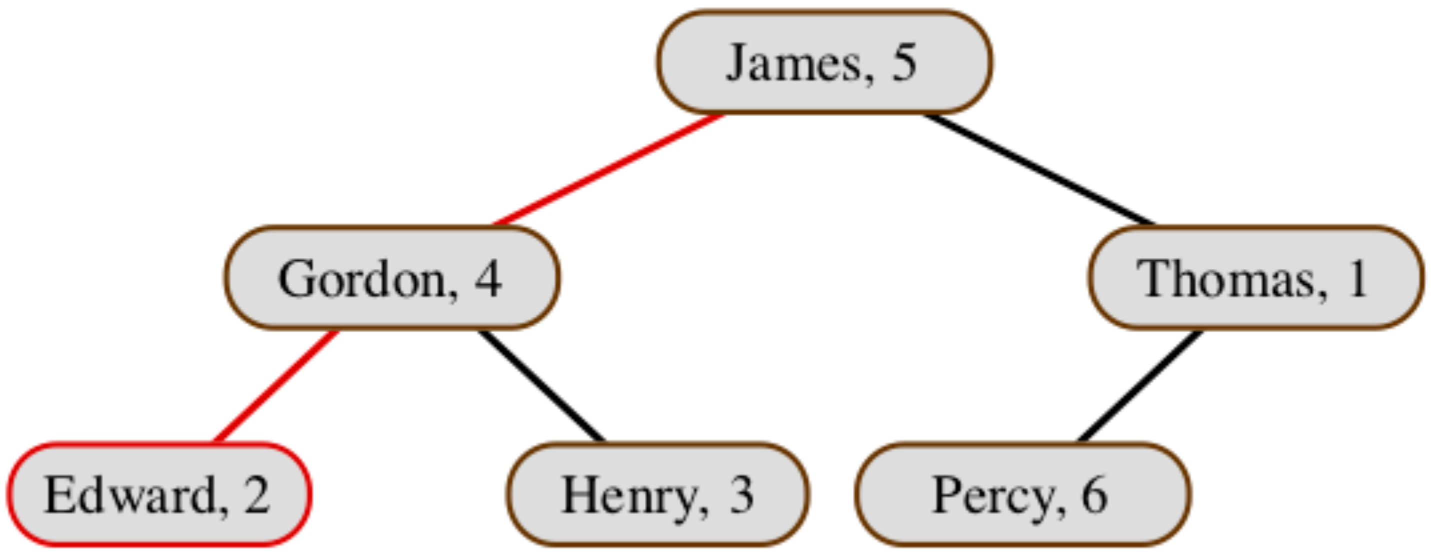
Binary Search Trees

- Use **binary trees** as a more efficient representation than a list to get a better lookup complexity.



Binary Search Trees

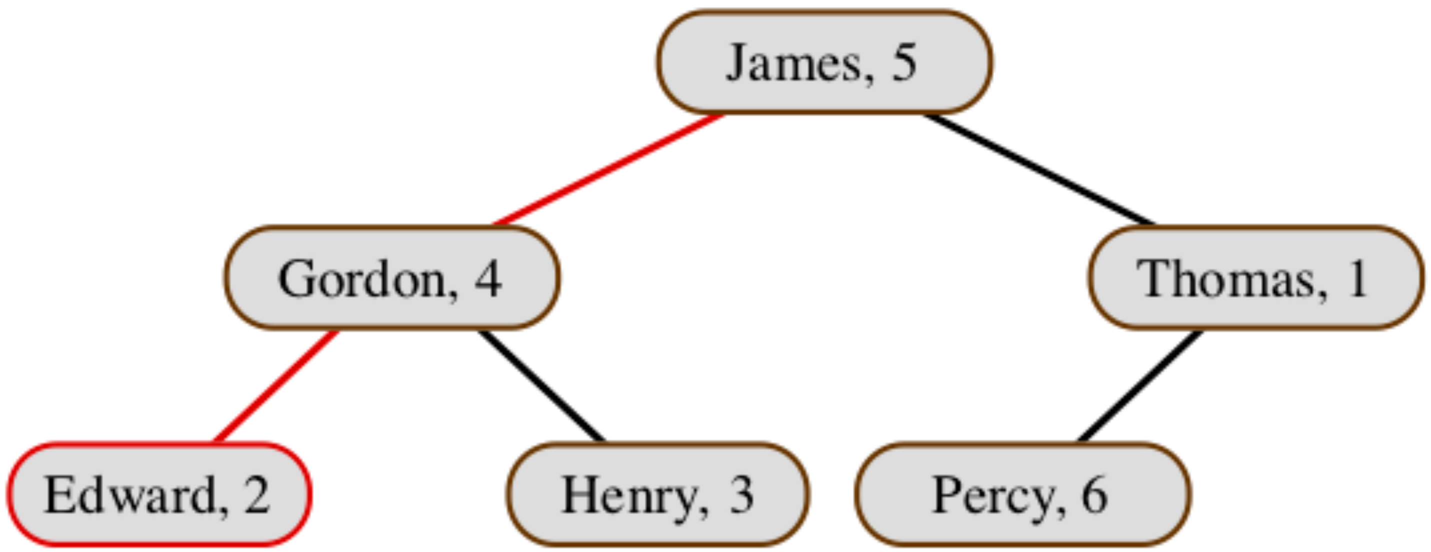
- Use **binary trees** as a more efficient representation than a list to get a better lookup complexity.



- Each node holds a `(key, value)` with a total ordering for the keys
- The *left* subtree holds smaller keys and the *right* subtree holds larger keys

Binary Search Trees

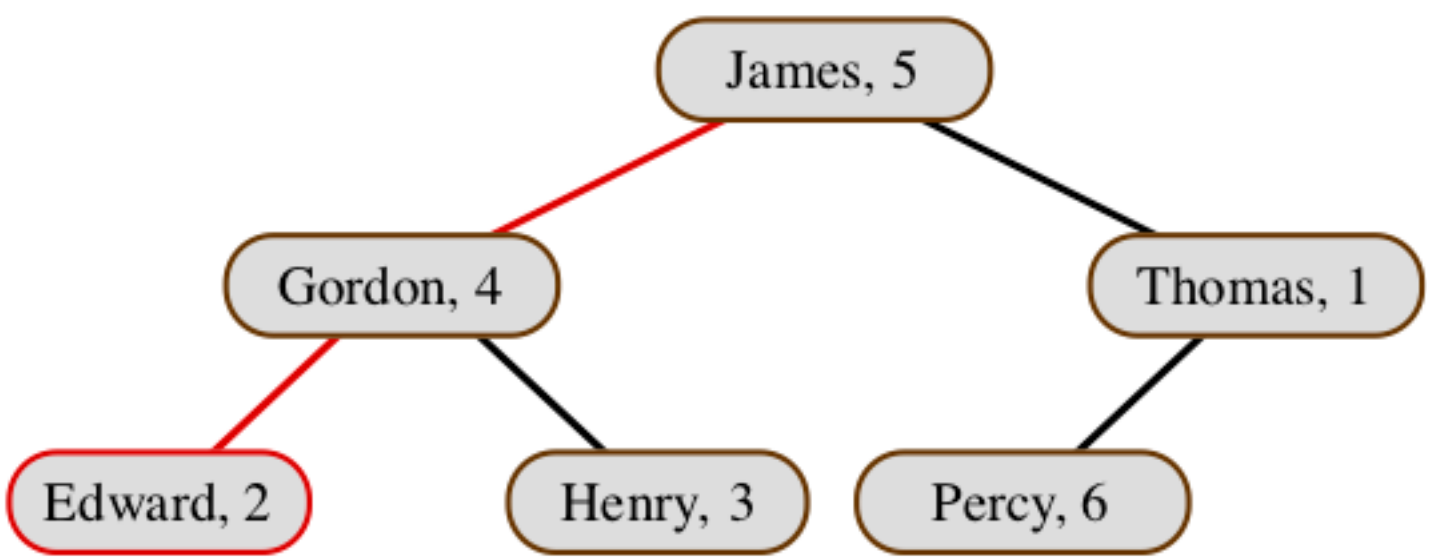
- Use **binary trees** as a more efficient representation than a list to get a better lookup complexity.



- If *balanced* then lookup is $O(\log n)$
- If *unbalanced* then lookup can be $O(n)$

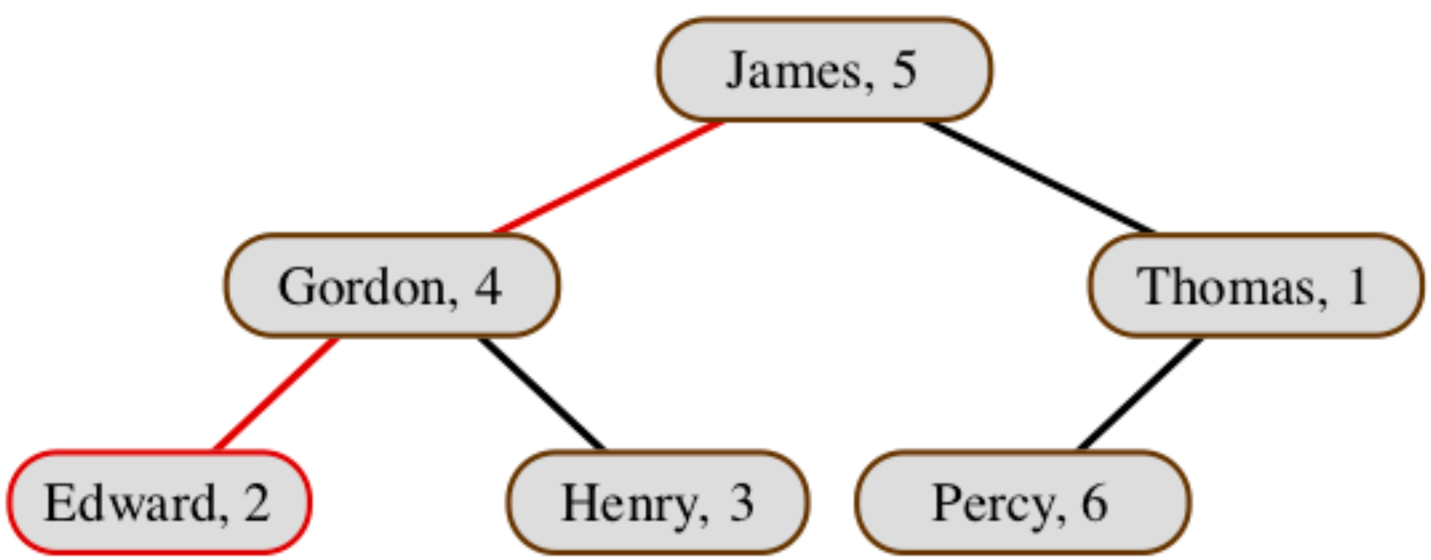
Binary Search Trees

```
# exception Missing of string
exception Missing of string

# let rec lookup = function
  | Br ((a, x), t1, t2), b ->
      if b < a then
        lookup (t1, b)
      else if a < b then
        lookup (t2, b)
      else
        x
  | Lf, b -> raise (Missing b)
val lookup : (string * 'a) tree * string -> 'a = <fun>
```



Binary Search Trees

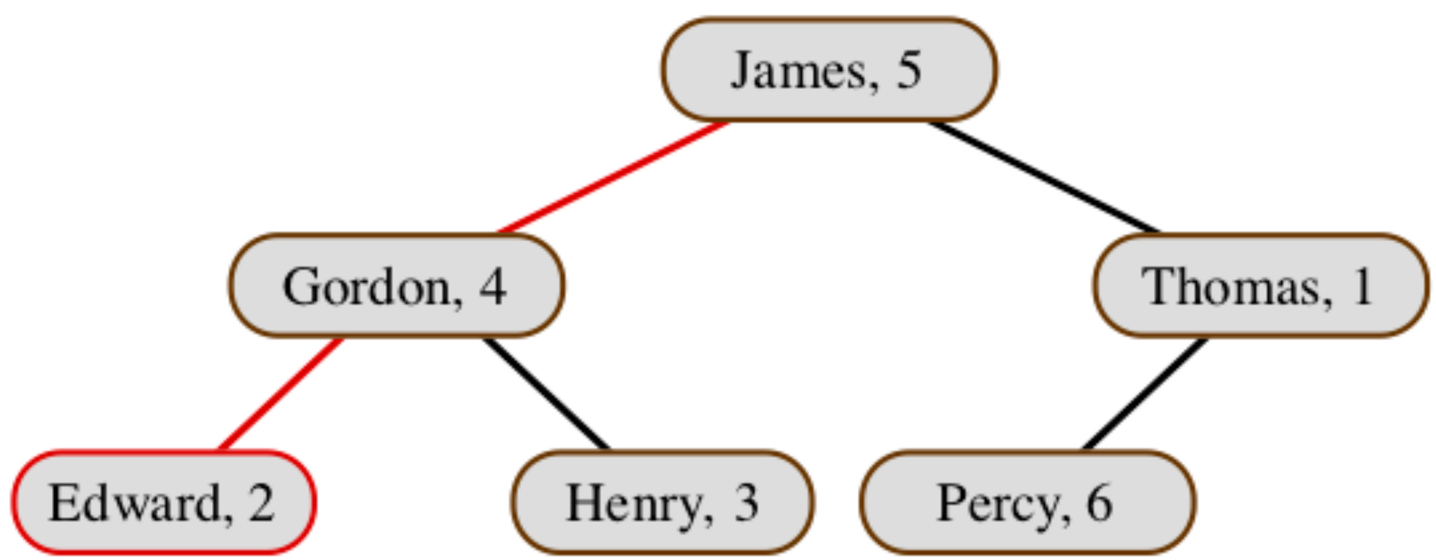
```
# exception Missing of string
exception Missing of string

# let rec lookup = function
  | Br ((a, x), t1, t2), b ->
      if b < a then
        lookup (t1, b)
      else if a < b then
        lookup (t2, b)
      else
        x
  | Lf, b -> raise (Missing b)
val lookup : (string * 'a) tree * string -> 'a = <fun>
```



Binary Search Trees

```
# exception Missing of string
exception Missing of string

# let rec lookup = function
  | Br ((a, x), t1, t2), b ->
      if b < a then
        lookup (t1, b)
      else if a < b then
        lookup (t2, b)
      else
        x
  | Lf, b -> raise (Missing b)
val lookup : (string * 'a) tree * string -> 'a = <fun>
```

O(log n) if the tree is balanced

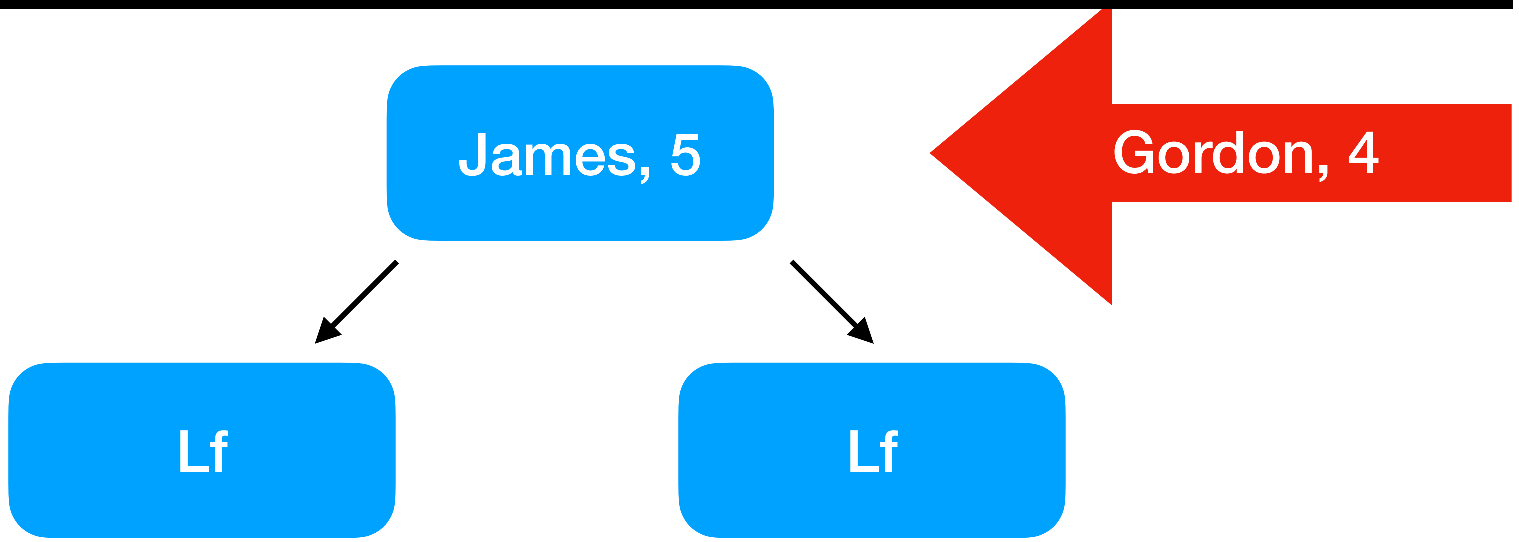


Binary Search Trees

```
# let rec update k v = function
  | Lf -> Br ((k, v), Lf, Lf)
  | Br ((a, x), t1, t2) ->
      if k < a then
        Br ((a, x), update k v t1, t2)
      else if a < k then
        Br ((a, x), t1, update k v t2)
      else (* a = k *)
        Br ((a, v), t1, t2)

val update : 'a -> 'b -> ('a * 'b) tree -> ('a * 'b) tree = <fun>
```

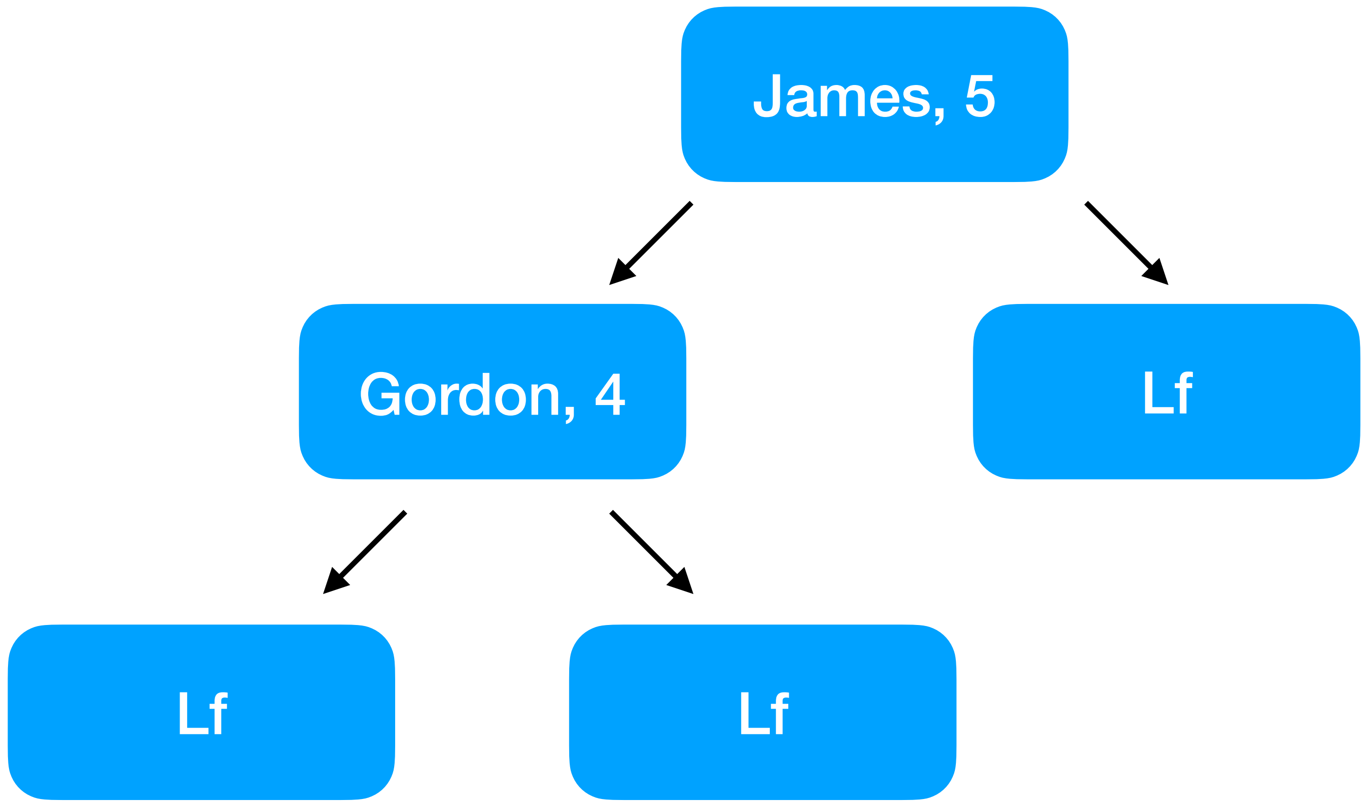
Binary Search Trees

```
# let rec update k v = function
    | Lf -> Br ((k, v), Lf, Lf)
    | Br ((a, x), t1, t2) ->
        if k < a then
          Br ((a, x), update k v t1, t2)
        else if a < k then
          Br ((a, x), t1, update k v t2)
        else (* a = k *)
          Br ((a, v), t1, t2)

val update : 'a -> 'b -> ('a * 'b) tree -> ('a * 'b) tree = <fun>
```



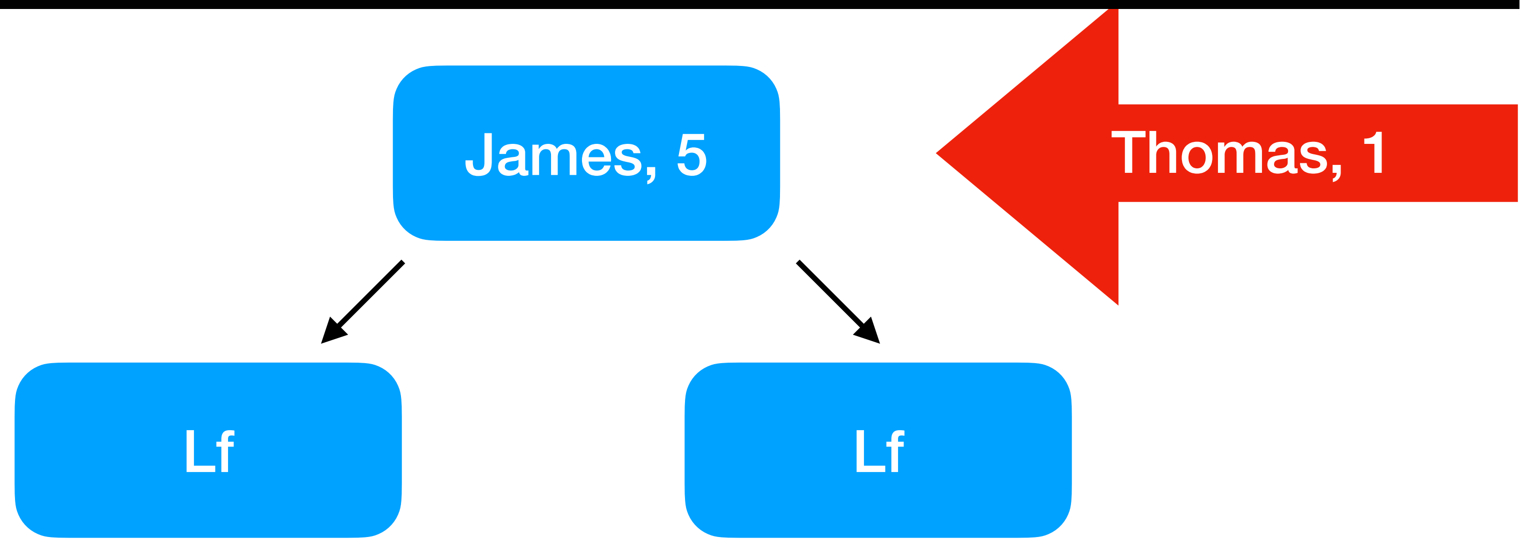
Binary Search Trees

```
# let rec update k v = function
    | Lf -> Br ((k, v), Lf, Lf)
    | Br ((a, x), t1, t2) ->
        if k < a then
          Br ((a, x), update k v t1, t2)
        else if a < k then
          Br ((a, x), t1, update k v t2)
        else (* a = k *)
          Br ((a, v), t1, t2)

val update : 'a -> 'b -> ('a * 'b) tree -> ('a * 'b) tree = <fun>
```



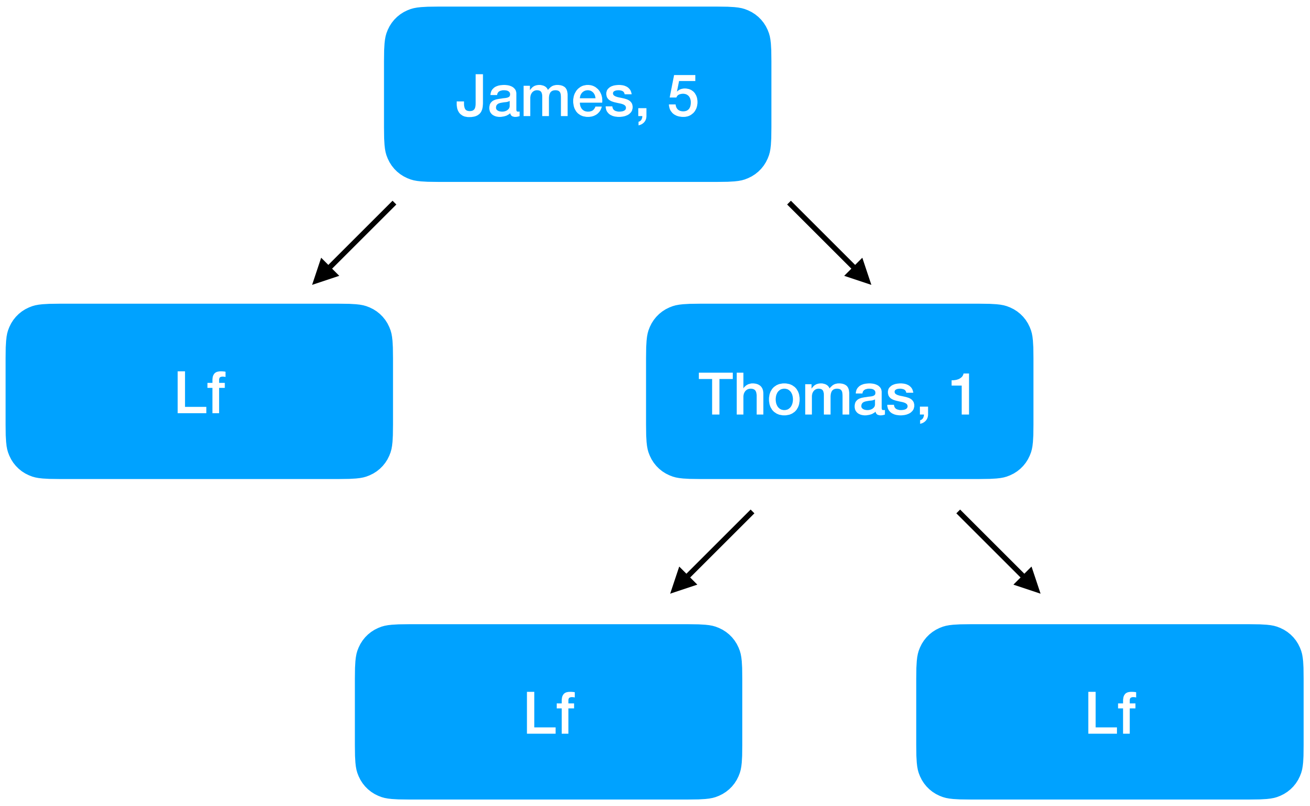
Binary Search Trees

```
# let rec update k v = function
    | Lf -> Br ((k, v), Lf, Lf)
    | Br ((a, x), t1, t2) ->
        if k < a then
          Br ((a, x), update k v t1, t2)
        else if a < k then
          Br ((a, x), t1, update k v t2)
        else (* a = k *)
          Br ((a, v), t1, t2)

val update : 'a -> 'b -> ('a * 'b) tree -> ('a * 'b) tree = <fun>
```



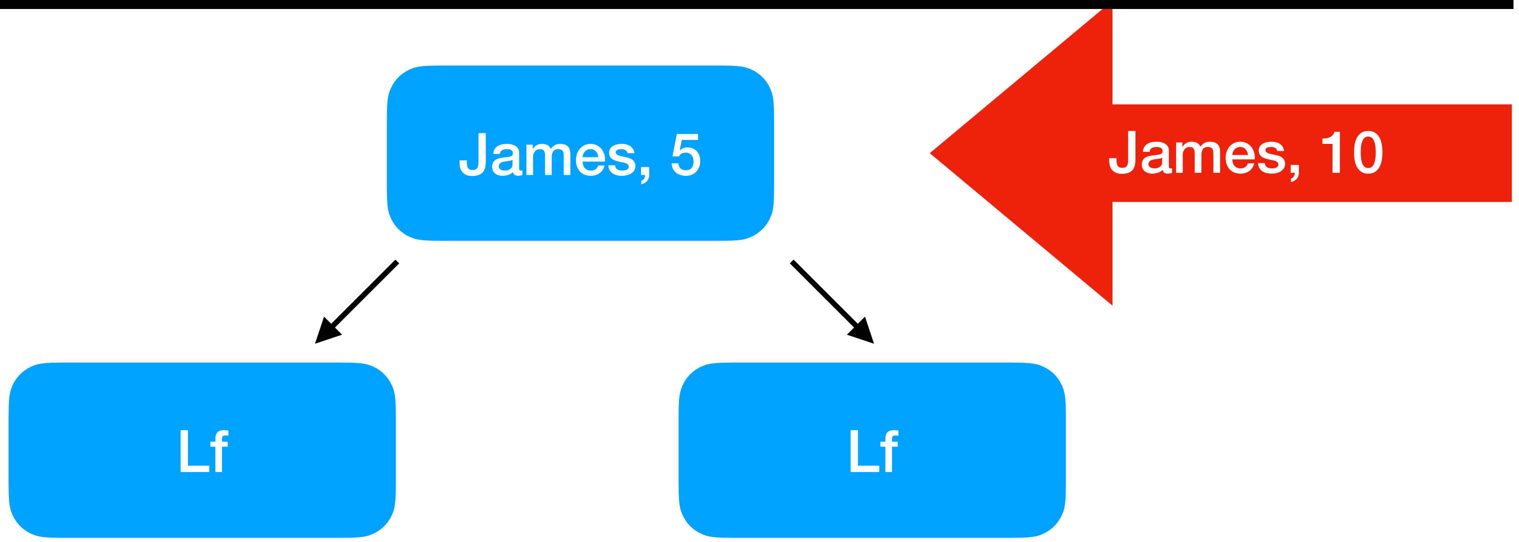
Binary Search Trees

```
# let rec update k v = function
  | Lf -> Br ((k, v), Lf, Lf)
  | Br ((a, x), t1, t2) ->
      if k < a then
        Br ((a, x), update k v t1, t2)
      else if a < k then
        Br ((a, x), t1, update k v t2)
      else (* a = k *)
        Br ((a, v), t1, t2)

val update : 'a -> 'b -> ('a * 'b) tree -> ('a * 'b) tree = <fun>
```



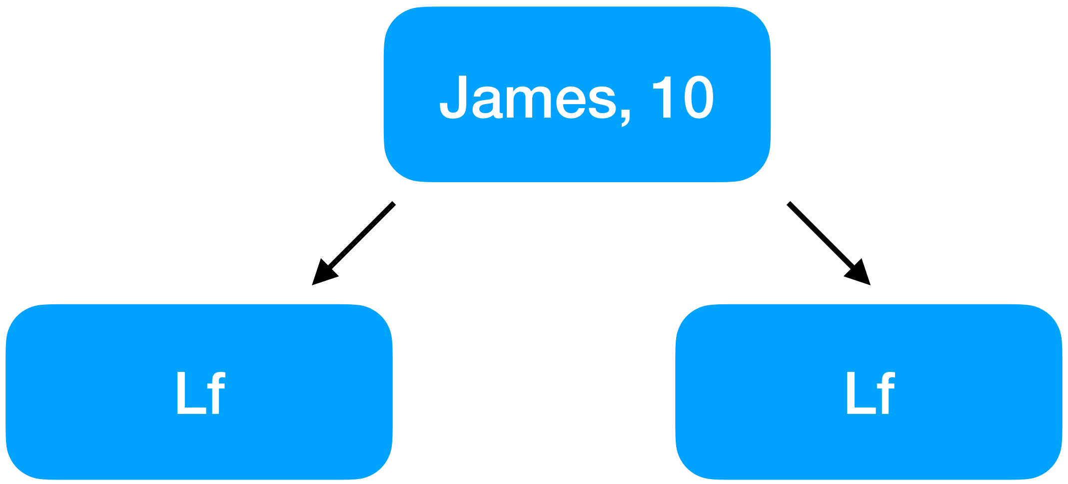
Binary Search Trees

```
# let rec update k v = function
    | Lf -> Br ((k, v), Lf, Lf)
    | Br ((a, x), t1, t2) ->
        if k < a then
          Br ((a, x), update k v t1, t2)
        else if a < k then
          Br ((a, x), t1, update k v t2)
        else (* a = k *)
          Br ((a, v), t1, t2)

val update : 'a -> 'b -> ('a * 'b) tree -> ('a * 'b) tree = <fun>
```



Binary Search Trees

```
# let rec update k v = function
  | Lf -> Br ((k, v), Lf, Lf)
  | Br ((a, x), t1, t2) ->
      if k < a then
        Br ((a, x), update k v t1, t2)
      else if a < k then
        Br ((a, x), t1, update k v t2)
      else (* a = k *)
        Br ((a, v), t1, t2)

val update : 'a -> 'b -> ('a * 'b) tree -> ('a * 'b) tree = <fun>
```



Binary Search Trees

```
# let rec update k v = function
  | Lf -> Br ((k, v), Lf, Lf)
  | Br ((a, x), t1, t2) ->
      if k < a then
        Br ((a, x), update k v t1, t2)
      else if a < k then
        Br ((a, x), t1, update k v t2)
      else (* a = k *)
        Br ((a, v), t1, t2)

val update : 'a -> 'b -> ('a * 'b) tree -> ('a * 'b) tree = <fun>
```

- We *reconstruct* the part of the structure that has changed and return the *updated* version.

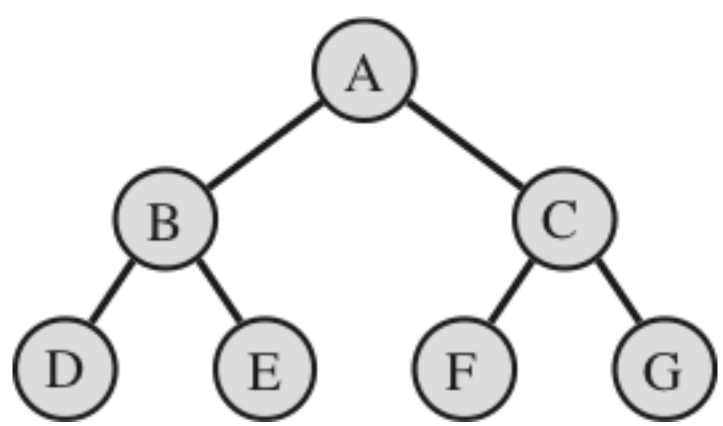
- OCaml *shares* the original structure, and values pointing to the original remain unchanged.

- This is also known as a *persistent data structure.*

Traversing Trees

Tree traversal refers to visiting the nodes of each tree in a well-defined order.

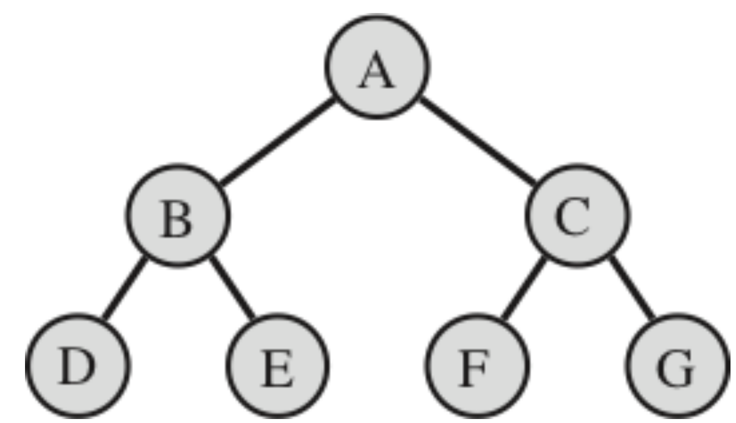
- **preorder** visits the label first (`ABDECFG`)
- **inorder** visits the label midway (`DBEAFCG`)
- **postorder** visits the label last (`DEBFGCA`)



Traversing Trees: preorder

- **preorder** visits the label first (`ABDECFG`)

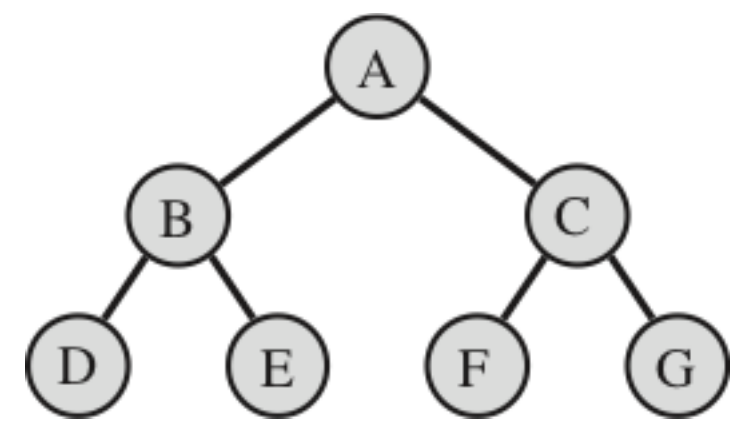
```
# let rec preorder = function
  | Lf -> []
  | Br (v, t1, t2) ->
      [v] @ preorder t1 @ preorder t2

val preorder : 'a tree -> 'a list = <fun>
```



Traversing Trees: inorder

- **inorder** visits the label midway (`DBEAFCG`)

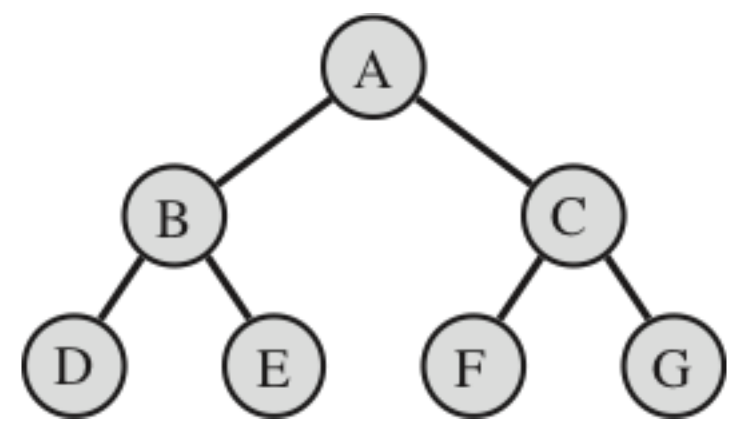
```
# let rec inorder = function
  | Lf -> []
  | Br (v, t1, t2) ->
      inorder t1 @ [v] @ inorder t2

val inorder : 'a tree -> 'a list = <fun>
```



Traversing Trees: inorder

- **inorder** visits the label midway (`DBEAFCG`)

```
# let rec inorder = function
  | Lf -> []
  | Br (v, t1, t2) ->
      inorder t1 @ [v] @ inorder t2

val inorder : 'a tree -> 'a list = <fun>
```



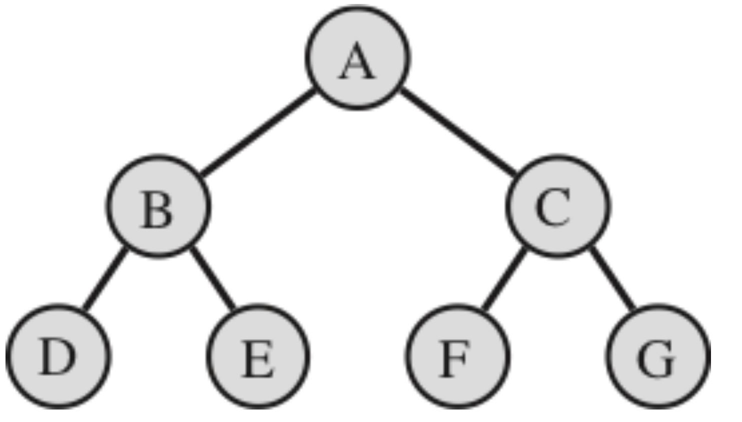
For binary search trees, this order respects the sorting constraint (left key < right key)

Also imaginatively known as a **treesort**.

Traversing Trees: postorder

- **postorder** visits the label last (`DEBFGCA`)

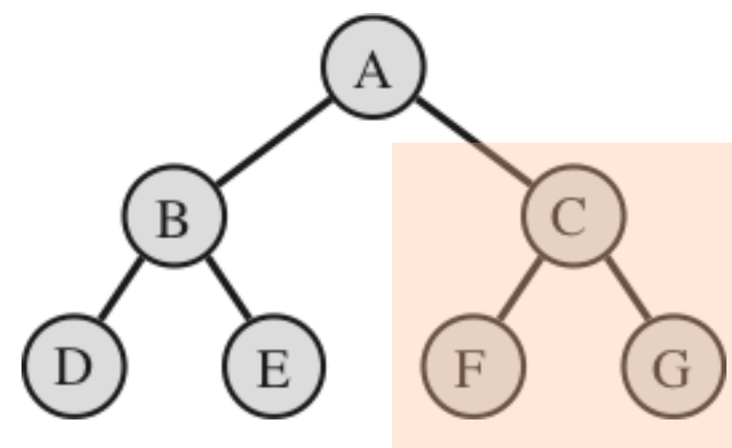
```
# let rec postorder = function
  | Lf -> []
  | Br (v, t1, t2) ->
      postorder t1 @ postorder t2 @ [v]

val postorder : 'a tree -> 'a list = <fun>
```



Traversing Trees: postorder

- **postorder** visits the label last (`DEBFGCA`)

```
# let rec postorder = function
  | Lf -> []
  | Br (v, t1, t2) ->
      postorder t1 @ postorder t2 @ [v]

val postorder : 'a tree -> 'a list = <fun>
```

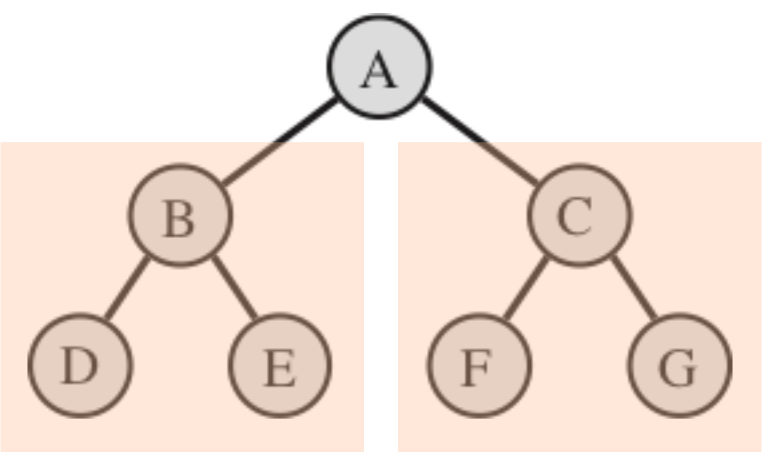


FGC

Traversing Trees: postorder

- **postorder** visits the label last (`DEBFGCA`)

```
# let rec postorder = function
  | Lf -> []
  | Br (v, t1, t2) ->
      postorder t1 @ postorder t2 @ [v]

val postorder : 'a tree -> 'a list = <fun>
```

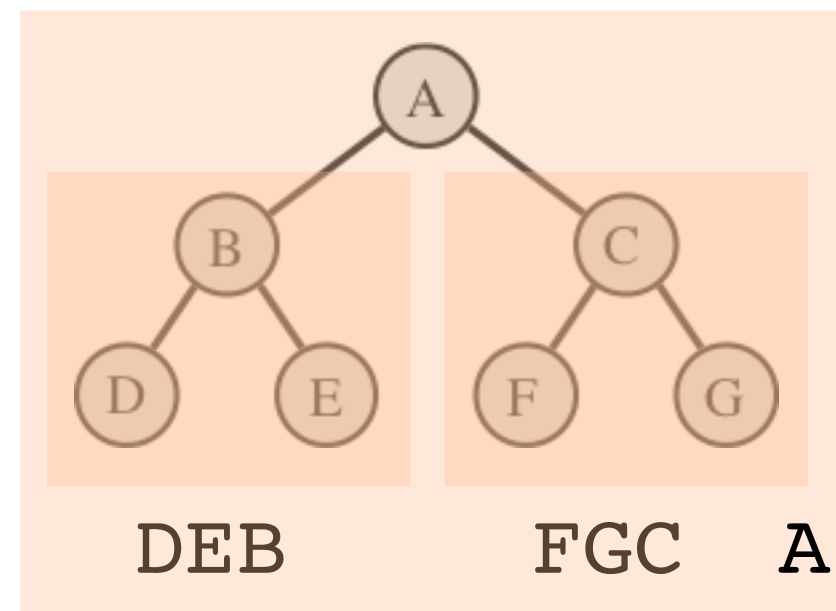


DEB FGC

Traversing Trees: postorder

- **postorder** visits the label last (`DEBFGCA`)

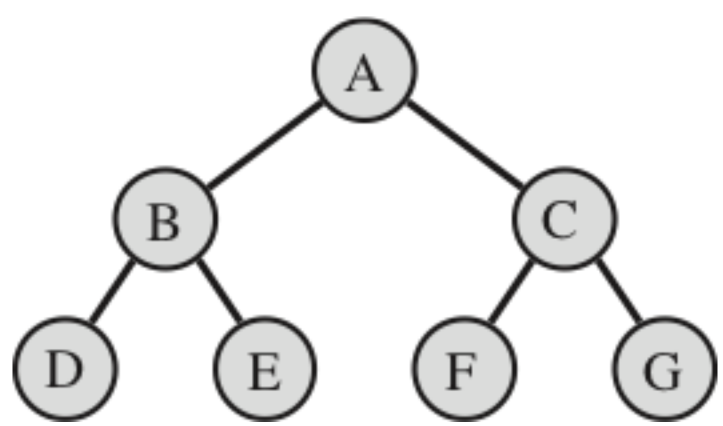
```
# let rec postorder = function
  | Lf -> []
  | Br (v, t1, t2) ->
      postorder t1 @ postorder t2 @ [v]

val postorder : 'a tree -> 'a list = <fun>
```



Traversing Trees

Tree traversal refers to visiting the nodes of each tree in a well-defined order.

- preorder, inorder and postorder are **depth-first** traversal algorithms.
- The other possibility is **breadth-first** by going across the levels of the tree.



Arrays

Arrays are an indexed storage area for values

- Very common data structure alongside lists and trees in most languages.
- Arrays are usually updated *in-place* and are *imperative* or *mutable* data structures.
- Are used in many classic algorithms such as the original Hoare in-place partition-sort.

Arrays

Arrays are an indexed storage area for values

- Elements of a list can only be reached by counting from the head of the list.
- Elements of a tree can be reached by following a path from the root.
- Elements of an array are uniformly designated by number (the "subscript").

Functional Arrays

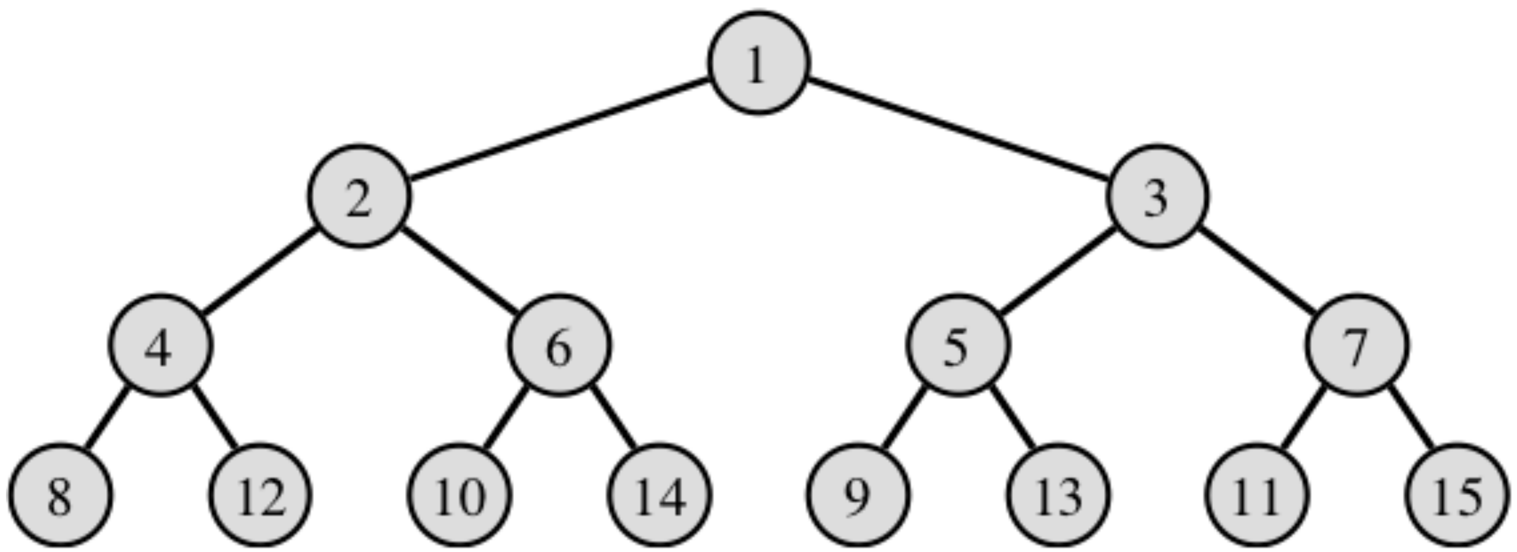
Arrays are an indexed storage area for values

Let's first consider an immutable array

- This is known as a *functional array* that is a finite map from integers to data.
- Updating implies copying the array to return a new version, but pointers to old copies remain.
- Can updates be efficient?

Functional Trees

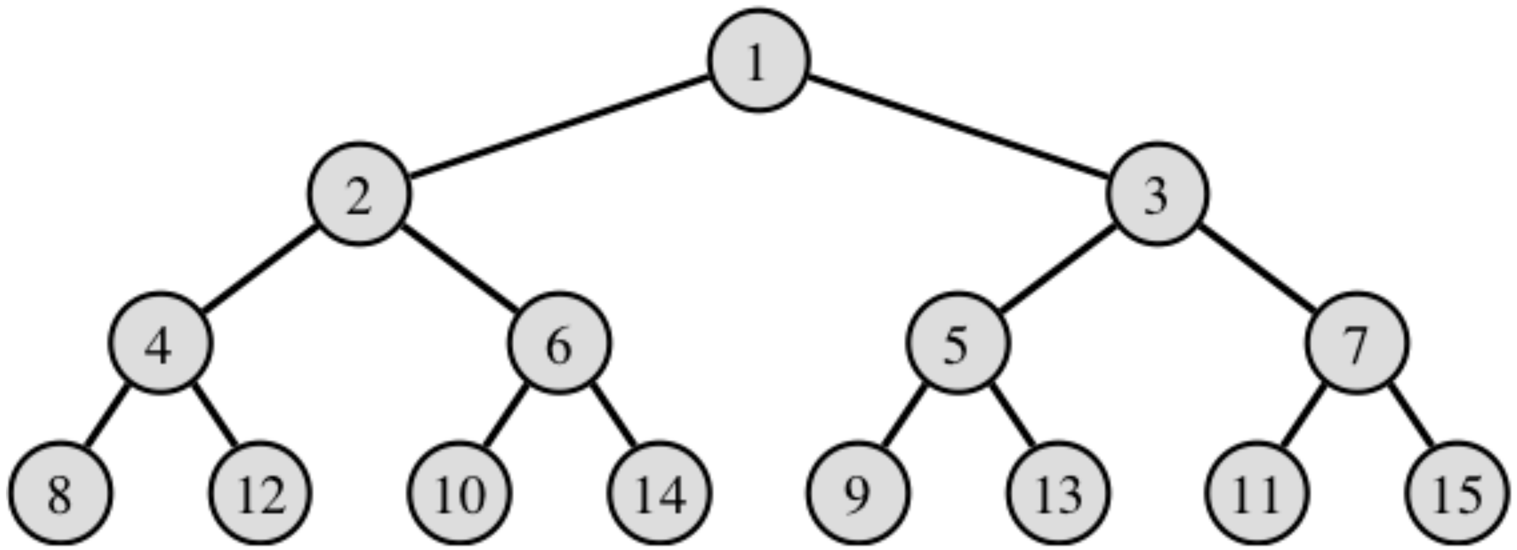
The path to element *i* follows the **binary code** for *i* (the "subscript")



- The numbers above are not the values, but the positions of array elements.
- Complexity of access to this is always $O(\log n)$ as the tree is always balanced.

Functional Trees

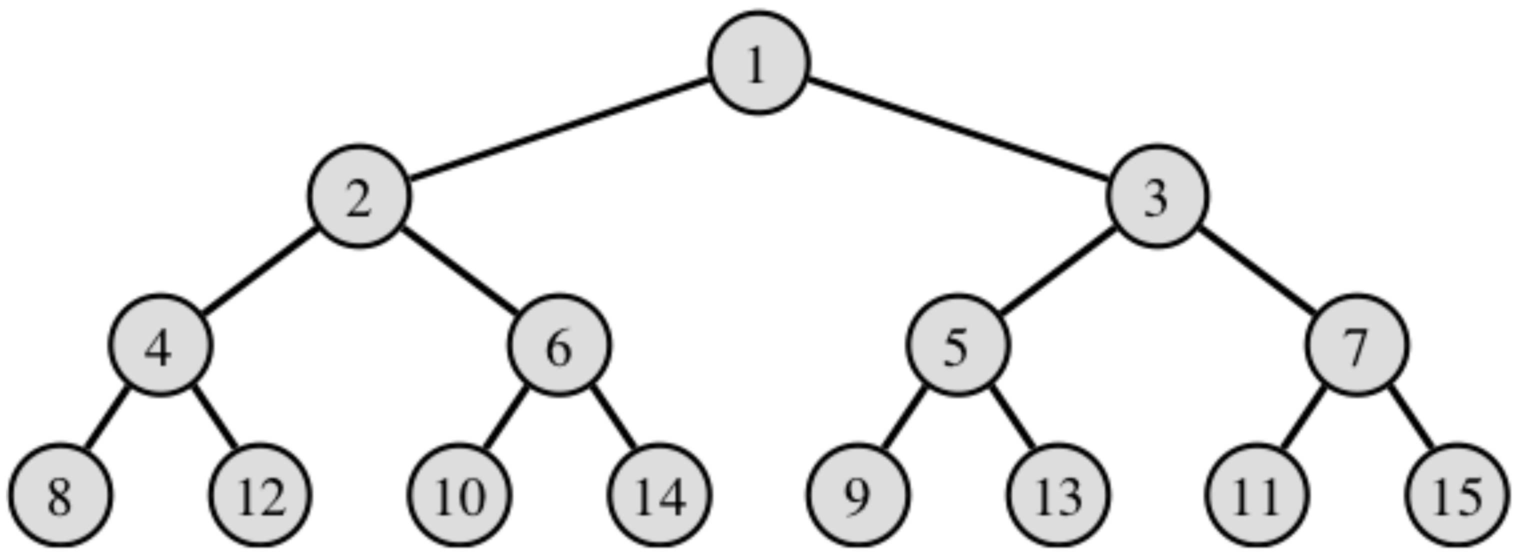
The path to element *i* follows the **binary code** for *i* (the "subscript")




```
# exception Subscript

# let rec sub = function
  | Lf, _ -> raise Subscript
  | Br (v, t1, t2), k ->
      if k = 1 then v
      else if k mod 2 = 0 then
        sub (t1, k / 2)
      else
        sub (t2, k / 2)
```

Functional Trees

The path to element *i* follows the **binary code** for *i* (the "subscript")




```
# exception Subscript

# let rec sub = function
  | Lf, _                                    -> raise Subscript
  | Br (v, t1, t2), 1                        -> v
  | Br (v, t1, t2), k when k mod 2 = 0       -> sub (t1, k / 2)
  | Br (v, t1, t2), k                        -> sub (t2, k / 2)
```

Functional Trees

The path to element *i* follows the **binary code** for *i* (the "subscript")

```
# let rec update = function
  | Lf, k, w ->
      if k = 1 then
        Br (w, Lf, Lf)
      else
        raise Subscript  (* Gap in tree *)
  | Br (v, t1, t2), k, w ->
      if k = 1 then
        Br (w, t1, t2)
      else if k mod 2 = 0 then
        Br (v, update (t1, k / 2, w), t2)
      else
        Br (v, t1, update (t2, k / 2, w))
```

Functional Trees

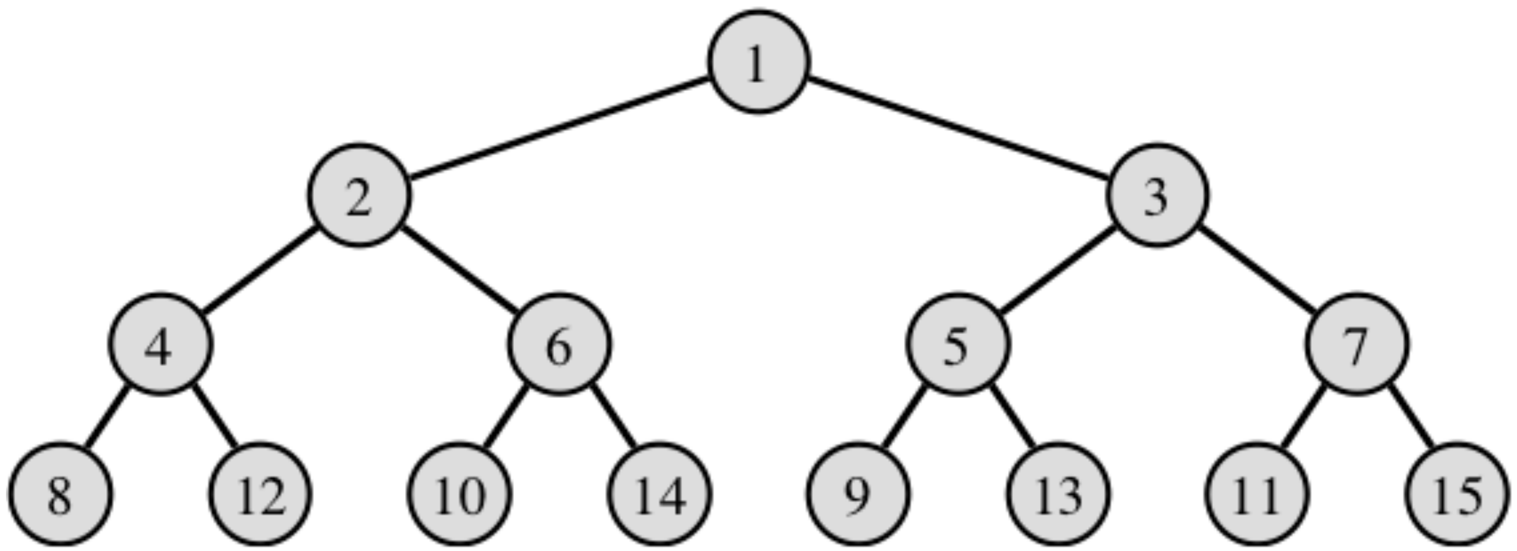
The path to element *i* follows the **binary cod** script")

O(log n) if the tree is balanced

```
# let rec update = function
    | Lf, k, w ->
        if k = 1 then
          Br (w, Lf, Lf)
        else
          raise Subscript  (* Gap in tree *)
    | Br (v, t1, t2), k, w ->
        if k = 1 then
          Br (w, t1, t2)
        else if k mod 2 = 0 then
          Br (v, update (t1, k / 2, w), t2)
        else
          Br (v, t1, update (t2, k / 2, w))
```

Functional Trees

The path to element *i* follows the **binary code** for *i* (the "subscript")



15 = 0b1111

12 = 0b1100

11 = 0b1011

Complexity of Dictionary Data Structures

- **Linear search:** Most general, needing only equality on keys, but inefficient (linear time).

- **Binary search:** Needs an ordering on keys. $O(\log n)$in the average case, binary search trees are $O(n)$ in the worst case.

- **Array subscripting:** Least general, requiring keys to be integers, but even worst-case time is $O(\log n)$.