BOOLEAN MATRICES

$(\{true, false\}, \vee, false, \wedge, true)$

$(m \times n)$ Boolean matrix.

$M = (M_{i,j})_{\substack{0 \le i < m \\ 0 \le j < n}}$ $\quad M_{i,j} \in \{true, false\}$

$M \longmapsto \underline{rel}(M) \subseteq [m] \times [n]$

$\overset{\text{def}}{=} \{(i,j) \in [m] \times [n] \mid M_{i,j} = true\}$.

$R \subseteq [m) \times [n)$

$\big\downarrow$

$\underline{\text{mat}}(R)$ $(m \times n)$ - Boolean matrix.

$\overset{\text{def}}{=}$

$\left(\underline{\text{mat}}(R)_{i,j}\right)_{\substack{0 \le i < m \\ 0 \le j < n}}$

$\underline{\text{mat}}(R)_{i,j} \overset{\text{def}}{=} \text{true} \iff (i,j) \in R$

show

$\underline{\text{mat}}(\underline{\text{rel}}(M))_{i,j} = M_{i,j}$

Claim:

Bijection

$M \longmapsto \underline{\text{rel}}(M) \longmapsto \underline{\text{mat}}(\underline{\text{rel}}(M)) = M$

$R \longmapsto \underline{\text{mat}}(R) \longmapsto \underline{\text{rel}}(\underline{\text{mat}}(R)) = R$

$$\frac{[m] \overset{R}{\longrightarrow\!\!\!\!\!\!+} [n] \overset{S}{\longrightarrow\!\!\!\!\!\!+} [\ell]}{[m] \overset{S \circ R}{\longrightarrow\!\!\!\!\!\!+} [\ell]}$$

$$\frac{\overset{M}{(m \times n)}\text{-Bool. matrix} \qquad \overset{N}{(n \times \ell)}\text{-Bool. matrix}}{N \circledast M \quad (m \times \ell) \text{ matrix}}$$

// def

$$(N \circledast M)_{i,j} =^{\text{def}} \bigvee_k \left(M_{i,k} \wedge N_{k,j} \right)$$

Claim:

$$\underline{mat}(R) \circledast \underline{mat}(S) = \underline{mat}(R \circ S)$$

M, N $(m \times n)$-Bool. mat.

$$(M \oplus N)_{i,j} =_{df} M_{i,j} \vee N_{i,j}$$

Claim:

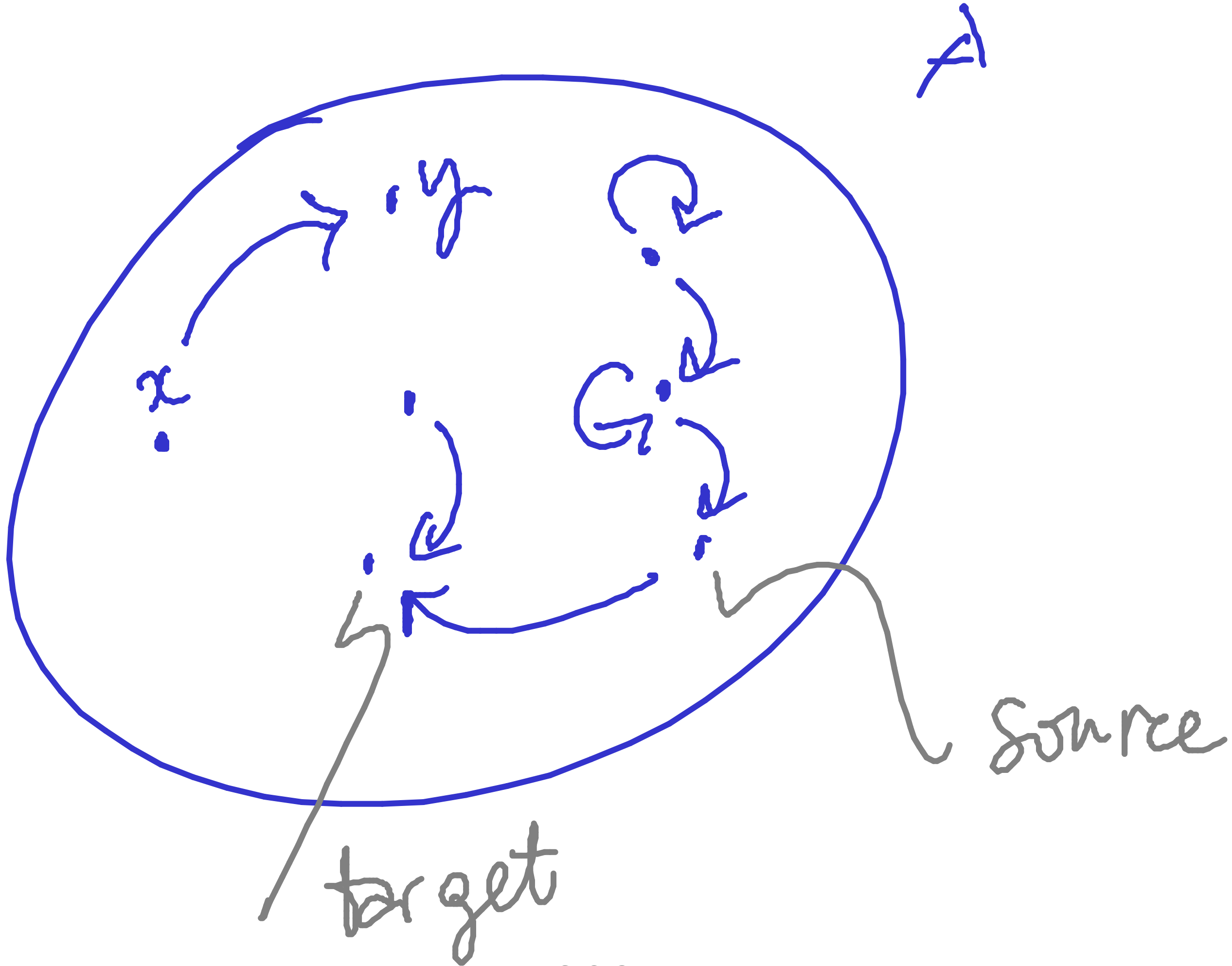
$$\underline{rel}(M \oplus N) = \underline{rel}(M) \cup \underline{rel}(N)$$

Relations from $[m]$ to $[n]$ and $(m \times n)$-matrices over Booleans provide two alternative views of the same structure.

This carries over to identities and to composition/multiplication .

Directed graphs

Definition 130 *A* directed graph (A, R) *consists of a set* A *and a relation* R *on* A *(i.e. a relation from* A *to* A*).*



Corollary 132 *For every set A, the structure*

$$\left(\underset{\substack{\| \mathrm{def} \\ \mathcal{P}(A \times A)}}{\mathrm{Rel}(A)} \ , \ \mathrm{id}_A \ , \circ \ \right)$$

is a monoid.

Definition 133 *For $R \in \mathrm{Rel}(A)$ and $n \in \mathbb{N}$, we let*

$$R^{\circ n} \ = \ \underbrace{R \circ \cdots \circ R}_{n \text{ times}} \ \in \mathrm{Rel}(A)$$

be defined as id_A for $n = 0$, and as $R \circ R^{\circ m}$ for $n = m + 1$.

Proposition 135 *Let* (A, R) *be a directed graph. For all* $n \in \mathbb{N}$ *and* $s, t \in A$, $s\ R^{\circ n}\ t$ *iff there exists a path of length* n *in* R *with source* s *and target* t.

PROOF: By induction:

Base case: $(n = 0)$

$s\ R^{\circ 0}\ t \overset{?}{\Longleftrightarrow} \exists$ path of len 0 from s to t

$\Updownarrow$ $\Updownarrow$

$s\ \mathrm{id}_A\ t \Longleftrightarrow s = t$

Inductive step: $(n \in \mathbb{N})$

$s\ R^{\circ n}\ t \Leftrightarrow \exists$ path of len n from s to t

RTP: $s\, R^{\circ(n+1)}\, t \overset{?}{\iff} \exists$ path of len. $n+1$ from s to t.

$\Updownarrow$

$s\,(R \circ R^{\circ n})\, t$

$\Updownarrow$

$\exists u.\ \underbrace{s\, R^{\circ n}\, u}_{} \wedge \underline{u\, R\, t} \iff \exists$ path of len. 1 from u to t

$\Updownarrow$ (IH)

$\exists$ path of len. n from s to u

$\boxtimes$

REFLEXIVE-TRANSITIVE CLOSURE.

Definition 136 *For* $R \in \mathrm{Rel}(A)$, *let*

$$R^{\circ *} = \bigcup \{ R^{\circ n} \in \mathrm{Rel}(A) \mid n \in \mathbb{N} \} = \bigcup_{n \in \mathbb{N}} R^{\circ n} \quad .$$

Corollary 137 *Let* (A, R) *be a directed graph. For all* $s, t \in A$, $s \; R^{\circ *} \; t$ *iff there exists a path with source* s *and target* t *in* R.

NB:

$$M_k = I_n + M + M^2 + \cdots + M^k$$

The $(n \times n)$-matrix $M = \mathrm{mat}(R)$ of a finite directed graph $([n], R)$ for n a positive integer is called its *adjacency matrix*.

The adjacency matrix $M^* = \mathrm{mat}(R^{\circ *})$ can be computed by matrix multiplication and addition as M_n where

$$\begin{cases} M_0 & = & I_n \\ M_{k+1} & = & I_n + (M \cdot M_k) \end{cases}$$

This gives an algorithm for establishing or refuting the existence of paths in finite directed graphs.

Partial functions

Definition 141 *A relation* $R : A \rightarrow\!\!\!\!\!+ B$ *is said to be* functional*, and called a* partial function*, whenever it is such that*

$$\forall a \in A. \forall b_1, b_2 \in B.\ a\,R\,b_1 \ \wedge\ a\,R\,b_2 \implies b_1 = b_2 \quad .$$

$\underline{\mathrm{PFun}}(A,B) \subseteq \underline{\mathrm{Rel}}(A,B) =_{\mathrm{def}} \mathcal{P}(A \times B)$

Examples:

- OCaml program.
- $(-)^{-1} : \mathbb{R} \to \mathbb{R}$ not defined at 0.

  $x \longmapsto x^{-1} \quad (x \neq 0)$

Theorem 143 *The identity relation is a partial function, and the composition of partial functions yields a partial function.*

NB

$$f = g : A \rightharpoonup B$$

iff

$$\forall a \in A. \big(f(a){\downarrow} \iff g(a){\downarrow} \big) \wedge f(a) = g(a)$$

<u>Notation</u>: $f : A \rightarrow B$ partial function

$f(a){\downarrow} \iff \exists b \in B.\ a \, f \, b$

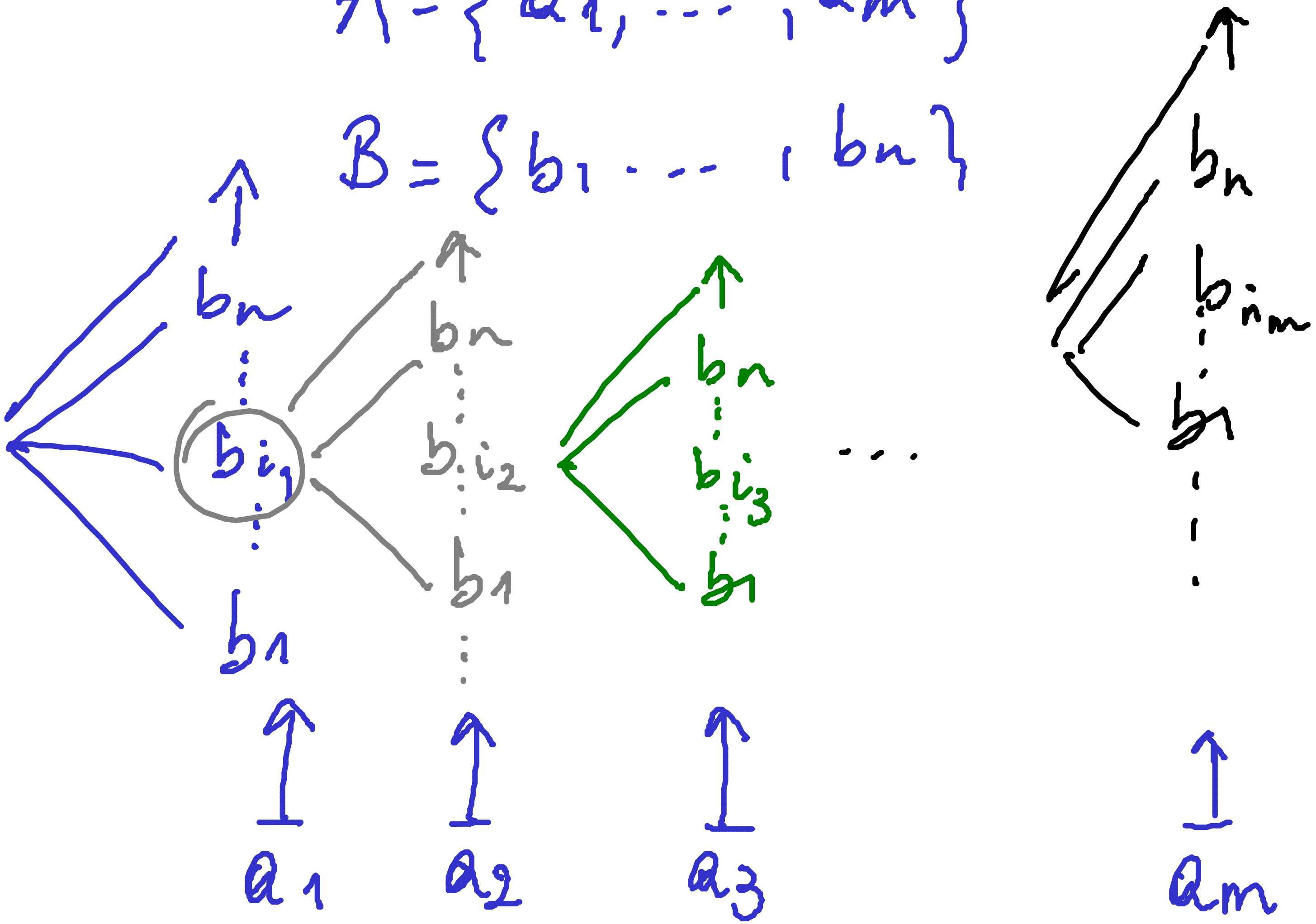
$f(a){\uparrow} \iff \neg\big(f(a){\downarrow} \big)$

Proposition 144 *For all finite sets* A *and* B,

$$\#(A \rightharpoonup B) = (\#B + 1)^{\#A} .$$

PROOF IDEA: $A = \{a_1, \ldots, a_m\}$

$B = \{b_1 \ldots, b_n\}$



Functions (or maps)

Definition 145 *A partial function is said to be* total*, and referred to as a* (total) function *or* map*, whenever its domain of definition coincides with its source.* $(A \Rightarrow B)$

$$= \qquad \underline{\mathrm{Fun}}(A,B) \subseteq \underline{\mathrm{PFun}}(A,B) \subseteq \underline{\mathrm{Rel}}(A,B)$$

the partial functions with outputs for all inputs.

<u>NB</u>: If f is a total function from A to B then $\forall a \in A$, $f(a)\downarrow$

Theorem 146 *For all* $f \in \mathrm{Rel}(A, B)$,

$$f \in (A \Rightarrow B) \iff \forall\, a \in A.\ \exists!\, b \in B.\ a \mathrel{f} b \quad .$$

Examples:

The set of Boolean $(m\times n)$-matrices

$$\begin{tikzcd} \underline{Rel}([m],[n]) \arrow[r, bend left, "\underline{mat}"] & (m\times n)\text{-}\underline{BoolMat} \arrow[l, bend left, "\underline{rel}"] \end{tikzcd}$$

functions.

$$\begin{cases} \underline{rel} \circ \underline{mat} = \underline{id}_{\underline{Rel}([m],[n])} \\ \underline{mat} \circ \underline{rel} = \underline{id}_{(m\times n)\text{-}\underline{BoolMat}} \end{cases}$$

Bijections

Proposition 147 *For all finite sets* A *and* B,

$$\#(A \Rightarrow B) = \#B^{\#A} .$$

Exponentiation.

PROOF IDEA:

Theorem 148 *The identity partial function is a function, and the composition of functions yields a function.*

NB

1. $f = g : A \to B$ iff $\forall\, a \in A.\, f(a) = g(a)$.

2. For all sets A, the identity function $\mathrm{id}_A : A \to A$ is given by the rule

   $$\mathrm{id}_A(a) = a$$

   and, for all functions $f : A \to B$ and $g : B \to C$, the composition function $g \circ f : A \to C$ is given by the rule

   $$\big(g \circ f\big)(a) = g\big(f(a)\big) \quad .$$